

# RECURRENT ORBITS OF SUBGROUPS OF LOCAL COMPLEX ANALYTIC DIFFEOMORPHISMS

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## ABSTRACT

We show recurrent phenomena for orbits of groups of local complex analytic diffeomorphisms that have a certain subgroup or image by a morphism of groups that is non-virtually solvable. In particular we prove that a non-virtually solvable subgroup of local biholomorphisms has always recurrent orbits, i.e. there exists an orbit contained in its set of limit points.

## 1. INTRODUCTION

We are interested in the interaction between algebraic properties of groups of local complex analytic diffeomorphisms and their topological dynamics. We denote by  $\text{Diff}(\mathbb{C}^n, 0)$  the group of local complex analytic diffeomorphisms defined in a neighborhood of the origin of  $\mathbb{C}^n$ . The algebraic nature of subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$  is studied from different points of view in the literature, for instance in the context of groups of real analytic diffeomorphisms in compact manifolds [6, Ghys], the existence of faithful analytic actions of mapping class groups of surfaces on surfaces [5, Cantat-Cerveau], the study of integrability properties of one-dimensional foliations [11, Rebelo-Reis] [4, Câmara-Scardua], local intersection dynamics [16, Seigal-Yakovenko] [1, Binyamini], the study of the derived length [14, Martelo-Ribón] [13]...

This paper generalizes two results of Rebelo and Reis [12] (Theorems 1 and 2) that relate algebraic properties of a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^2, 0)$  (being non-virtually solvable) with the existence of non-discrete orbits for the action of  $G$  or more rigorously for the action of any pseudogroup that is a representative of  $G$ . The biggest generalization is that our results hold for every dimension.

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Let us introduce the main results in the paper. Theorem 1 provides a series of sufficient conditions for the non-discreteness of a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ .

**Theorem 1.** *Let  $G$  be a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  such that at least one of the following conditions holds:*

- (1)  $G$  is a non-solvable unipotent subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ .
- (2)  $G$  is a non-virtually solvable subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  consisting of non-hyperbolic elements.
- (3)  $\overline{(j^1G)}_0$  is a non-solvable subgroup of  $\text{GL}(n, \mathbb{C})$ .
- (4)  $j^1G$  is not virtually reducible and  $\overline{(j^1G)}_0 \not\subset \mathbb{C}^* \text{Id}$ .
- (5)  $j^1G$  is not virtually reducible and the group induced by  $\overline{j^1G}$  in  $\text{PGL}(n, \mathbb{C})$  is non-discrete.

*Then there exists a finite subset  $\mathcal{S}$  of  $G$  such that for every pseudogroup  $\mathcal{P}$  that is a representative of  $\langle \mathcal{S} \rangle$  there exist a connected open neighborhood  $V$  of 0 and a sequence  $(f_n : V \rightarrow f_n(V))_{n \geq 1}$  in  $\mathcal{P} \setminus \{\text{Id}_V\}$  converging uniformly to  $\text{Id}$  in  $V$ . In particular all the points in  $V$  except at most a countable union of proper analytic sets are recurrent for the action of  $\mathcal{P}$ .*

Let us explain the previous theorem. We say that a local diffeomorphism  $\phi \in \text{Diff}(\mathbb{C}^n, 0)$  is *unipotent* if its linear part  $D_0\phi$  at 0 is a unipotent linear automorphism of  $\mathbb{C}^n$ . We say that a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  is unipotent if all its elements are unipotent. A unipotent group of local diffeomorphisms is “well-represented” by a Lie algebra of nilpotent formal vector fields. As a consequence problems in unipotent subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$  can be interpreted as simpler problems in their Lie algebras.

Condition (2) requires that  $G$  has no finite index solvable subgroup. Moreover the definition of hyperbolic local diffeomorphism is not the most common one, we say that  $\phi \in \text{Diff}(\mathbb{C}^n, 0)$  is *hyperbolic* if  $\text{spec}(D_0\phi)$  is not contained in  $\mathbb{S}^1$ . Regarding Condition (3),  $\overline{j^1G}$  is the closure of  $j^1G := \{D_0\phi : \phi \in G\}$  in  $\text{GL}(n, \mathbb{C})$  in the usual topology. It is well-known that  $\overline{j^1G}$  is a Lie group and Condition (3) is imposed in its connected component  $\overline{(j^1G)}_0$  of  $\text{Id}$ . Conditions (4) and (5) will be discussed later on.

The thesis of the theorem is that we have a finite subset  $\{g_1, \dots, g_p\}$  in  $G$  such that given any choice  $\phi_j : U_j \rightarrow V_j$  of biholomorphism such that the germ of  $\phi_j$  at 0 is equal to  $g_j$  for  $1 \leq j \leq p$ , we obtain the family  $(f_n)_{n \geq 1}$  by considering certain words on the symbols

$\{\phi_1, \dots, \phi_p, \phi_1^{-1}, \dots, \phi_p^{-1}\}$ . The biholomorphism  $f_n$  is defined in a domain that depends on the chosen word but anyway such domain contains  $V$  for any  $n \in \mathbb{N}$ . More precisely, the words providing the elements in the sequence  $(f_n)_{n \geq 1}$  are commutators obtained by applying the so called Zassenhaus lemma that will be discussed later on. This strategy was applied in dimension 2 by Rebelo and Reis [12]. We generalize it to any dimension.

Conditions (1) and (2) in Theorem 1 can be weakened by checking them out on all subgroups of  $G$ . For instance Theorem 1 holds if we replace Condition (1) with the existence of a non-solvable unipotent subgroup of  $G$ .

In order to introduce Theorem 2 let us define the discrete orbits property. Consider a pseudogroup  $\mathcal{P}$  of holomorphic maps, i.e. a family  $(f_j : U_j \rightarrow V_j)_{j \in J}$  of biholomorphisms closed by compositions, inverses, restrictions and patching (cf. Definition 2.9). We say that  $\mathcal{P}$  is a representative of a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  if  $0 \in U_j \cap V_j$ ,  $U_j \cup V_j \subset \mathbb{C}^n$ ,  $f_j(0) = 0$  for any  $j \in J$  and  $G$  is the group of germs at 0 of elements of  $(f_j)_{j \in J}$ . We say that  $G$  has discrete orbits (locally discrete orbits in [12]) if there exists a representative pseudogroup  $(f_j : U_j \rightarrow V_j)_{j \in J}$  of  $G$  such that all orbits  $\mathcal{O}(P) = \{f_j(P) : P \in U_j\}$  are discrete sets.

**Theorem 2.** *Let  $G$  be a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  with discrete orbits. Then  $G$  is virtually solvable.*

Given a non-virtually solvable subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  Theorem 2 provides an orbit  $\mathcal{O}$  without isolated points for every choice of a representative pseudogroup of  $G$ , i.e. for every choice of domains of definition of the elements of  $G$ . All points of  $\mathcal{O}$  are recurrent. Some parts of the proof are inspired in analogue results of [12] even if there are specific issues that are associated to the higher dimensional setting.

Theorems 1 and 2 were known in dimension 1 as a consequence of results of Shcherbakov [18] and Nakai [9]. Indeed given a non-solvable subgroup  $G$  of  $\text{Diff}(\mathbb{C}, 0)$  there exists a real analytic curve  $\Sigma$  such that any representative  $\mathcal{P}$  of  $G$  has dense orbits in the connected components of  $U \setminus \Sigma$  for some open neighborhood  $U$  of 0 [9]. Moreover there exist real flows of non-trivial holomorphic vector fields in the topological closure of any representative pseudogroup of  $G$ . Theorems 1 and 2 are not as precise, the density condition is replaced with the existence of recurrent orbits. Anyway, it is not clear how to generalize the density results to the higher dimensional setting. For example, consider the group  $H = \{\phi \in \text{Diff}(\mathbb{C}^n, 0) : f \circ \phi \equiv f\}$  for some germ  $f$  of non-constant holomorphic function defined in the neighborhood of 0 in  $\mathbb{C}^n$ . The group is non-solvable for  $n > 1$  since its Lie algebra (the set of

formal singular vector fields that have  $f$  as first integral) is non-solvable [14]. Analogously the subgroup of  $H$  of its tangent to the identity elements is non-solvable. Hence there are finitely generated subgroups of  $H$  that are non-solvable and consist only of tangent to the identity elements. The orbit of any point by the action of such a group  $J$  is contained in a level set of  $f$  and thus has empty interior. In spite of this, Theorem 1(1) applies to  $J$  providing recurrent points.

Every instance of Theorem 1 is related to the non-solvability of the connected component of  $Id$  of a certain group. In items (1) and (2) the conditions are equivalent to the non-solvability of  $G \cap \overline{G}_0^z$ . Let us remark that  $\overline{G}^z$  is the Zariski-closure of  $G$  (cf. section 4.2) and is obtained as a projective limit of algebraic matrix groups. The group  $\overline{G}_0^z$  is its connected component of  $Id$ , that is generated by the exponential of the Lie algebra of  $\overline{G}^z$  (cf. [14]). The condition in item (3) applies to  $j^1G$  and it is equivalent to the non-solvability of  $j^1G \cap \overline{(j^1G)}_0$ . Hence the conditions in items (1), (2) and (3) are analogous once we consider the proper definition of Zariski-closure for subgroups of local diffeomorphisms. Conditions (4) and (5) imply Condition (3) and as a consequence all the hypotheses are part of a common framework.

Let us introduce an example illustrating the previous discussion. Consider a subgroup  $G$  of  $SL(2, \mathbb{C})$  that induces a non-abelian free Kleinian subgroup of  $PSL(2, \mathbb{C})$  whose limit set  $\Lambda$  is not the whole Riemann sphere. We can even suppose that  $G$  is free. Let  $U$  be the dense open set consisting of points whose directions belong to  $\hat{\mathbb{C}} \setminus \Lambda$ . The orbits through points in  $U$  are discrete. Therefore  $G$  is highly non-trivial (it is free!) but all the connected components of  $Id$  associated to subgroups of  $G$  in items (1)-(5) of Theorem 1 are solvable.

In a slightly different point of view the paper can be interpreted as a study on how non-virtual solvability impacts the dynamics of a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ . Indeed the hypotheses in Theorem 1 imply that some group canonically associated to  $G$  is not virtually solvable. This is obvious for Condition (2). Since Conditions (4) and (5) imply Condition (3), it suffices to show that we can replace non-solvable with non-virtually solvable in Conditions (1) and (3) of Theorem 1. The group  $\overline{(j^1G)}_0$  is a connected Lie group and hence it is solvable if and only if it is virtually solvable. Analogously every virtually solvable unipotent subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  is solvable. Non-virtually solvable groups are part of the dichotomy provided by the Tits alternative: given a field  $F$  of characteristic 0 every subgroup of the group  $GL(n, F)$  is either virtually solvable or contains a non-abelian free subgroup [19].

Unfortunately the Tits alternative does not hold a priori for subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$ . Anyway the elements of the sequence  $(f_n)_{n \geq 1}$  in Theorem 1 are obtained by considering certain commutators of finitely many elements whose linear parts are close to  $Id$ . This process is specially simple for a non-abelian free group. Hence the proof of Theorem 1 is easier in a setting in which the Tits alternative holds. The cases (3), (4) and (5) in Theorem 1 are of Tits type, meaning that there exists a non-abelian free subgroup of  $j^1G$  on two generators whose linear parts can be chosen arbitrarily close to  $Id$  and hence we can apply the commutator process. As a consequence we can suppose in these cases that  $\mathcal{S}$  has two elements. The localization of generators of free groups in a neighborhood of the identity profits of results of Breuillard and Gelfander on the topological Tits alternative [3].

**1.1. Pseudo-solvable groups.** Let us explain how to deal with Condition (1) in Theorem 1 where we can not use the Tits alternative. More precisely, we can not guarantee that the hypothesis implies the existence of a non-abelian free subgroup of  $G$ . We will use the notion of pseudo-solvable subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  as a workaround for this issue.

The definition of pseudo-solvable group was introduced by Ghys in [6]. It is intended to identify whether or not a group is solvable by analyzing a finite generator set in specific cases, for instance in geometrical problems. Ghys uses this concept to construct sequences of non-trivial real analytic diffeomorphisms in real analytic manifolds that converge uniformly to the identity map. He explains that the definition of pseudo-solvable is arbitrary and many others are possible [6][p. 171]. We use in this paper one of the possible alternative definitions. More precisely we introduce the definition of  $p$ -pseudo-solvable group for  $p \in \mathbb{N} \cup \{0\}$  generalizing Ghys definition that corresponds to 1-pseudo-solvable. Our definition is intended to take profit of the particular algebraic structure of solvable subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$ .

**Definition 1.1.** Let  $G$  be a finitely generated group. Consider a finite set  $\mathcal{S}$  of generators of  $G$ . Fix  $p \in \mathbb{Z}_{\geq 0}$ . By recurrence we define  $\mathcal{S}_p(0) = \mathcal{S}$  and

$$\mathcal{S}_p(j+1) = \{[f, g] \text{ or } [g, f]; f \in \mathcal{S}_p(j), g \in \cup_{k=j-p}^j \mathcal{S}_p(k)\}.$$

We say that  $G$  is  $p$ -pseudo-solvable for  $\mathcal{S}$  if there exists  $j \in \mathbb{N}$  such that  $\mathcal{S}_p(j) = \{1\}$ . We say that  $G$  is  $p$ -pseudo-solvable if it is  $p$ -pseudo-solvable for some finite generator set. We drop the subindex  $p$  when it is implicit.

Let us clarify some points in the previous definition. It could happen that  $j - p < 0$ ; in such a case we consider  $\mathcal{S}(k) = \{1\}$  for any  $k < 0$ .

We always consider that given  $f \in \mathcal{S}(0)$  its inverse  $f^{-1}$  also belongs to  $\mathcal{S}(0)$ . Moreover since  $[f, g]^{-1} = [g, f]$  this condition holds for  $\mathcal{S}(j)$  for every  $j \geq 0$ .

The Zassenhaus lemma implies that if  $\mathcal{S}$  consists of elements close to  $Id$  then the elements in  $\mathcal{S}_p(k)$  tend uniformly to  $Id$  when  $k \rightarrow \infty$ . This idea was introduced by Ghys for the study of groups of real analytic diffeomorphisms of compact manifolds [6] and was adapted to the local setting by Loray and Rebelo [8] (cf. also [12]). Therefore non- $p$ -pseudo-solvable groups with generators close to  $Id$  provide sequences of non-trivial elements converging to the identity map. The condition of being close to the identity map is somehow automatic for the elements of a unipotent subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  and as a consequence in order to prove Theorem 1 in Case (1) it suffices to show that  $G$  is not  $p$ -pseudo-solvable for some  $p \in \mathbb{Z}_{\geq 0}$ . We complete the proof of Theorem 1 by showing that there exists  $p = p(n)$  such that a unipotent subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  is solvable if and only if is  $p$ -pseudo-solvable (Theorem 4.1). Let us remark that Rebelo and Reis show in their proof of the analogue of Theorem 1(1) for dimension 2 that 1-pseudo-solvable implies solvable for  $n = 2$  [12]. We show a weaker property, easier to prove and that anyway suffices to prove Theorems 1 and 2.

The definition of pseudo-solvable is useful in a class of groups if pseudo-solvable is equivalent to solvable; otherwise it is a property that depends on the choice of generators of the group and with no clear algebraic or geometrical meaning. The main drawback of the definition of pseudo-solvable is that the proof of this equivalence is in general quite technical. Let us explain how working with  $p$ -pseudo-solvability simplifies the proof.

Let  $G$  be a unipotent subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ , for example a group of tangent to the identity diffeomorphisms. We want to show that if a finitely generated subgroup of  $G$  is  $p$ -pseudo-solvable, for some  $p = p(n)$  to be determined later on, then  $G$  is solvable. Let  $m \in \mathbb{N}$  be the minimum integer such that  $\mathcal{S}(m) = \{Id\}$ . We define  $\mathcal{S}(j, k) = \cup_{l=j}^k \mathcal{S}(l)$  for  $0 \leq j \leq k$ ,  $\Gamma(j) = \langle \mathcal{S}(j, m) \rangle$  and  $G(j, j+p) = \langle \mathcal{S}(j, j+p) \rangle$ . It is clear that  $\Gamma(m)$  is solvable and the proof relies on proving that if  $\Gamma(j+1)$  is solvable then  $\Gamma(j)$  is solvable for any  $0 \leq j < m$ . Given  $\phi \in \mathcal{S}(j)$  and  $0 \leq q \leq p$  we have  $\phi \circ \psi \circ \phi^{-1} \circ \psi^{-1} \in \mathcal{S}(j+q+1)$  for any  $\psi \in \mathcal{S}(j+q)$  by definition of  $\mathcal{S}(j+q+1)$ . In particular we obtain

$$(1) \quad \phi G(j+1, j+q) \phi^{-1} \subset G(j+1, j+q+1)$$

for  $1 \leq q \leq p$ . If  $G(j+1, j+q) = G(j+1, j+q+1)$  then  $\phi$  normalizes  $G(j+1, j+q)$  and this provides valuable information about the

elements of  $\mathcal{S}(j)$  that can be used to show that  $\Gamma(j)$  is solvable. Unfortunately the previous groups can be different. Anyway we associate an invariant taking finitely many values to the subgroups in the increasing sequence

$$(2) \quad G(j+1, j+1) \subset G(j+1, j+2) \subset \dots \subset G(j+1, j+p+1)$$

of subgroups of the solvable group  $\Gamma(j+1)$ . Moreover the invariant is increasing in a lexicographical order. if  $p$  is big enough we can always find some  $1 \leq q \leq p$  such that the invariants associated to  $G(j+1, j+q)$  and  $G(j+1, j+q+1)$  coincide. The groups can be still different but we find solvable extensions of their Lie algebras (and even solvable extensions of the groups) that coincide and continue to satisfy the normalizing equation (1). The value of  $p$  can be chosen as the number of different values of the invariant. Our method associates an increasing sequence of solvable Lie algebras of formal vector fields to the sequence (2) and then relies on the classification of such Lie algebras in [14].

Theorems 1 and 2 illustrate how the description of the nature of an algebraic object, namely a Lie algebra of formal vector fields, entails dynamical consequences, and specifically recurrence properties, for unipotent subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$ .

Resuming we show a maximal-like condition on unipotent solvable subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$ . It allows to show  $p$ -pseudo-solvable  $\implies$  solvable by turning subnormalizing group properties into normalizing properties.

## 2. PRELIMINARIES AND NOTATIONS

Let us introduce some concepts that will be used in the paper.

**2.1. Groups and Lie algebras.** Let  $G$  be a group. Let us define the derived groups of  $G$ .

**Definition 2.1.** We define the *derived group*  $G^{(1)}$  (or  $[G, G]$ ) of  $G$  as the group generated by the commutators  $[f, g] := fgf^{-1}g^{-1}$  of elements of  $G$ . Analogously we define  $G^{(j+1)} = [G^{(j)}, G^{(j)}]$  for  $j \geq 1$ . We denote  $G^{(0)} = G$ .

The previous definition can be extended to any Lie algebra  $\mathfrak{g}$  by replacing the commutator with the Lie bracket.

**Definition 2.2.** We say that a group  $G$  (resp. Lie algebra) is *solvable* if there exists  $j \in \mathbb{N} \cup \{0\}$  such that  $G^{(j)}$  is trivial. In such a case we define its *derived length*  $\ell(G)$  as the minimum  $j \in \mathbb{N} \cup \{0\}$  such that



$G^{(j)}$  is trivial. We say that the derived length of a non-solvable group  $G$  (resp. Lie algebra) is equal to  $\infty$ .

The statements of Theorems 1 and 2 involve properties of finite index subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$  or  $\text{GL}(n, \mathbb{C})$ .

**Definition 2.3.** Consider a group property  $P$  (for example being abelian, free...). We say that a group  $G$  is *virtually*  $P$  if there exists a finite index subgroup of  $G$  that satisfies  $P$ .

*Remark 2.1.* Every finite index subgroup  $H$  of a group  $G$  contains a finite index normal subgroup of  $G$ . Indeed the intersection of all the conjugates of  $H$  is a finite index normal subgroup of  $G$ . The group properties that we consider in this paper are subgroup-closed. As a consequence given a virtually  $P$  group  $G$  there exists a finite index normal subgroup of  $G$  that has property  $P$ .

**2.2. Local diffeomorphisms.** We introduce the group of linear parts of a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ .

**Definition 2.4.** Let  $\phi \in \text{Diff}(\mathbb{C}^n, 0)$ . We denote by  $D_0\phi$  (or  $j^1\phi$ ) its differential at the origin.

**Definition 2.5.** Let  $G$  be a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ . We define the subgroup  $j^1G = \{j^1\phi : \phi \in G\}$  of  $\text{GL}(n, \mathbb{C})$ .

Let us define unipotent diffeomorphisms and groups. They are the objects of Theorem 4.1, one of the main results of the paper.

**Definition 2.6.** We say that a diffeomorphism  $\phi \in \text{Diff}(\mathbb{C}^n, 0)$  is *unipotent* if  $D_0\phi$  is a unipotent isomorphism, i.e.  $D_0\phi - Id$  is nilpotent. We say that  $\phi$  is *tangent to the identity* if  $D_0\phi \equiv Id$ .

**Definition 2.7.** We denote by  $\text{Diff}_u(\mathbb{C}^n, 0)$  the subset of  $\text{Diff}(\mathbb{C}^n, 0)$  consisting of unipotent diffeomorphisms. We say that a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  is *unipotent* if  $G \subset \text{Diff}_u(\mathbb{C}^n, 0)$ .

**Definition 2.8.** We define  $\text{Diff}_1(\mathbb{C}^n, 0)$  as the subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  consisting of tangent to the identity elements.

**2.3. Lie groups.** Let us see that the commutator map defined in a Lie group has attracting behavior in a neighborhood of the identity element. It will be useful to study matrix groups.

*Remark 2.2* (Zassenhaus lemma). Let  $T$  be a Lie group. Consider a left-invariant Riemannian metric defined in  $T$ . The map  $\theta : T \times T \rightarrow T$  defined by  $\theta(f, g) = [f, g]$  is differentiable and  $\theta|_{\{Id\} \times T} \equiv Id \equiv \theta|_{T \times \{Id\}}$ . Thus its differential at  $Id$  is equal to the identity map. Moreover there



exists a small neighbourhood  $V$  of  $Id$  in  $T$  and a constant  $C > 0$  such that

$$|[f, g] - Id| \leq C||f - Id|||g - Id| \quad \forall f, g \in V.$$

There exists  $\epsilon > 0$  such that  $W := \{f \in T : ||f - Id|| < \epsilon\}$  is contained in  $V$  and  $\epsilon < 1/(2C)$ . In particular we have  $|[f, g] - Id| < \epsilon/2$  for all  $f, g \in W$ .

The lemma implies that a discrete Lie group generated by elements sufficiently close to the identity is nilpotent (cf. [10][p. 147]).

**2.4. Pseudogroups.** We study the dynamics of pseudogroups induced by subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$ . First we introduce the definition of pseudogroups of homeomorphisms.

**Definition 2.9.** Consider a family  $\mathcal{F} = \{f_j : U_j \rightarrow V_j\}_{j \in J}$  of homeomorphisms where  $U_j, V_j$  are open sets of a topological space  $M$  for any  $j \in J$ . We say that  $\mathcal{F}$  is a *pseudogroup* on  $M$  if

- The identity map  $Id : U \rightarrow U$  belongs to  $\mathcal{F}$  for any open subset  $U$  of  $M$ .
- The map  $f_j^{-1} : V_j \rightarrow U_j$  belongs to  $\mathcal{F}$  for any  $j \in J$ .
- Given elements  $f_j : U_j \rightarrow V_j$  and  $f_k : U_k \rightarrow V_k$  of  $\mathcal{F}$  such that  $V_j \cap U_k \neq \emptyset$  the composition  $f_k \circ f_j : f_j^{-1}(V_j \cap U_k) \rightarrow f_k(V_j \cap U_k)$  belongs to  $\mathcal{F}$ .
- Given an element  $f : U \rightarrow V$  of  $\mathcal{F}$  and a non-empty open subset  $U'$  of  $U$ , the restriction  $f|_{U'} : U' \rightarrow f(U')$  belongs to  $\mathcal{F}$ .
- Let  $f : U \rightarrow V$  be a homeomorphism obtained by patching elements of  $\mathcal{F}$ , i.e. there exists an open covering  $\{U_j\}_{j \in J'}$  of  $U$  for some subset  $J'$  of  $J$  such that  $f|_{U_j} \equiv f_j$  for any  $j \in J'$ . Then  $f$  belongs to  $\mathcal{F}$ .

*Remark 2.3.* In the previous definition  $f|_{U_j}$  is considered as a map from  $U_j$  to  $f(U_j)$  and not as a map from  $U_j$  to  $V$ . Moreover if we say that an embedding  $f : U \rightarrow V$  belongs to a pseudogroup we mean that  $f|_U : U \rightarrow f(U)$  belongs to the pseudogroup. We will use this conventions for the sake of simplicity.

**Definition 2.10.** Consider a family  $\mathcal{F} = \{f_j : U_j \rightarrow V_j\}_{j \in J}$  of homeomorphisms as in Definition 2.9. We say that  $\mathcal{P}$  is the pseudogroup generated by  $\mathcal{F}$  if  $\mathcal{P}$  is the minimal pseudogroup defined in  $M$  and containing  $\mathcal{F}$ .

Let us explain how to obtain pseudogroups induced by a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$ . Roughly speaking we choose domains of definition for the elements of  $G$  (or for the elements of a generator set).

**Definition 2.11.** Consider a connected open neighborhood  $U$  of 0 in  $\mathbb{C}^n$ . Let  $\mathcal{F} = \{f_j : U_j \rightarrow V_j\}_{j \in J}$  be a family of biholomorphisms such that  $0 \in U_j \cap V_j$ ,  $U_j \cup V_j \subset U$ ,  $U_j$  is connected and  $f_j(0) = 0$  for any  $j \in J$ . We say that the pseudogroup  $\mathcal{P}$  generated by  $\mathcal{F}$  on  $U$  is induced by a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  if the group  $\langle \phi_j : j \in J \rangle$  is equal to  $G$  where  $\phi_j$  is the germ of  $f_j$  at 0.

*Remark 2.4.* The pseudogroup induced by a subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$  is not unique.

### 3. THE COMMUTATOR PROPERTY

Consider a finitely generated subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  that is not  $p$ -pseudo-solvable for  $\mathcal{S}$ . In this section we see that if the elements of  $\mathcal{S}$  have linear parts close to the identity map then there exists a sequence of non-trivial maps in  $G$  that converge uniformly to  $Id$  in some neighborhood of the origin. This property is crucial to obtain recurrent orbits. Rebelo and Reis showed analogue results in the case of groups that are not 1-pseudo-solvable [12]. The generalization is fairly straightforward and we include it for the sake of completeness.

**Definition 3.1.** Consider the usual norm defined in  $\mathbb{C}^n$ . Let  $\mathbb{B}_r^n$  be open the ball of center 0 and radius  $r$  in  $\mathbb{C}^n$ . Let  $f : \mathbb{B}_r^n \rightarrow \mathbb{C}^n$  be a holomorphic map. We define the norm of  $f$  in  $\mathbb{B}_r^n$  as

$$\|f\|_r = \sup_{(x_1, \dots, x_n) \in \mathbb{B}_r^n} |f(x_1, \dots, x_n)|.$$

Notice that  $\|A\|_1$  is the spectral norm for a  $n \times n$  matrix  $A$ .

Consider  $r, \epsilon, \tau > 0$  such that  $4\epsilon + \tau < r$ . Let  $f, g : \mathbb{B}_r^n \hookrightarrow \mathbb{C}^n$  be injective holomorphic maps such that  $\|f - Id\|_r \leq \epsilon$  and  $\|g - Id\|_r \leq \epsilon$ . Then the commutator  $[f, g] : \mathbb{B}_{r-4\epsilon}^n \hookrightarrow \mathbb{B}_r^n$  is defined in  $\mathbb{B}_{r-4\epsilon}^n$  and satisfies

$$(3) \quad \|[f, g] - Id\|_{r-4\epsilon-\tau} \leq \frac{2}{\tau} \|f - Id\|_r \|g - Id\|_r.$$

We will use Equation (3) with the elements of a generator set  $\mathcal{S}$  of a non-pseudo-solvable subgroup of  $\text{Diff}(\mathbb{C}^n, 0)$ . The previous estimate was introduced by Loray and Rebelo [8] (cf. also [12]). It is an analogue in the local setting of the Zassenhaus lemma (cf. Remark 2.2).

Next, let us see that if all elements of a generator set  $\mathcal{S}$  of a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  are close enough to the identity map then the elements of  $\mathcal{S}_p(j)$  tend uniformly to  $Id$  when  $j \rightarrow \infty$ .

**Proposition 3.1.** Fix  $p \in \mathbb{Z}_{\geq 0}$  and  $\delta > 0$  with  $(p+2)\delta < 1/4$ . Let  $\mathcal{S}$  be a subset of  $\text{Diff}(\mathbb{C}^n, 0)$  such that  $\mathcal{S} = \mathcal{S}^{-1}$ . Suppose  $g|_{\mathbb{B}_1^n}$  is a

well-defined injective holomorphic map such that  $\|g - Id\|_1 \leq \delta/4$  for any  $g \in \mathcal{S}$ . Then  $f|_{\mathbb{B}_{1/2}^n} : \mathbb{B}_{1/2}^n \rightarrow \mathbb{B}_1^n$  is an injective holomorphic map belonging to the pseudogroup generated by  $\{g|_{\mathbb{B}_{1-\delta/4}^n} : g \in \mathcal{S}\}$  on  $\mathbb{B}_1^n$  such that  $\|f - Id\|_{1/2} \leq \delta/2^{j+2}$  for all  $j \geq 0$  and  $f \in \mathcal{S}_p(j)$ .

The proof of Proposition 3.1 is split in the next lemmas.

**Lemma 3.1.** *Consider the hypotheses in Proposition 3.1. Let  $j \leq p+1$ . Then  $f$  is defined  $\mathbb{B}_{1-2j\delta}^n$  and satisfies  $\|f - Id\|_{1-2j\delta} \leq \delta/2^{j+2}$  for any  $f \in \mathcal{S}_p(j)$ .*

*Proof.* The proposition is obvious for  $j = 0$ . Let us show that if the result holds for  $0 \leq k \leq j < p+1$  then so it does for  $j+1$ .

An element of  $\mathcal{S}(j+1)$  is of the form  $[a, b]$  or  $[b, a]$  where  $a \in \mathcal{S}(j)$  and  $b \in \mathcal{S}(0, j)$ . Hence we obtain

$$\|a - Id\|_{1-2j\delta} \leq \delta/2^{j+2} \text{ and } \|b - Id\|_{1-2j\delta} \leq \delta/4.$$

We consider  $r = 1 - 2j\delta$ ,  $\epsilon = \delta/4$  and  $\tau = \delta$  in Equation (3). We get

$$\|[a, b] - Id\|_{1-2(j+1)\delta} \leq \frac{2}{\delta} \frac{\delta}{2^{j+2}} \frac{\delta}{4} = \frac{\delta}{2^{j+3}}$$

and the same inequality holds for  $[b, a]$ .  $\square$

**Lemma 3.2.** *Consider the hypotheses in Proposition 3.1. Let  $j > p+1$ . Then  $\|f - Id\|_{\kappa_j} \leq \delta/2^{j+2}$  for any  $f \in \mathcal{S}_p(j)$  where  $\kappa_j = 1 - \delta(2(p+1) + 1 + \frac{1}{2} + \dots + \frac{1}{2^{j-p-2}})$ .*

*Proof.* Let us show the result for  $j = p+2$ . Let  $f \in \mathcal{S}(p+2)$ ; it is of the form  $[a, b]$  or  $[b, a]$  where  $a \in \mathcal{S}(p+1)$  and  $b \in \mathcal{S}(1, p+1)$ . We consider  $r = 1 - 2(p+1)\delta$ ,  $\epsilon = \delta/8$  and  $\tau = \delta/2$ . We obtain

$$\|[a, b] - Id\|_{1-(2(p+1)+1)\delta} \leq \frac{4}{\delta} \frac{\delta}{2^{p+3}} \frac{\delta}{8} = \frac{\delta}{2^{p+4}}$$

and an analogous estimate holds for  $[b, a]$ .

Suppose that the result holds for any  $p+2 \leq k \leq j$  and let us prove that so it does for  $j+1$ . An element  $f$  of  $\mathcal{S}(j+1)$  is of the form  $[a, b]$  or  $[b, a]$  where  $a \in \mathcal{S}(j)$  and  $b \in \mathcal{S}(j-p, j)$ . We consider  $r = \kappa_j$ ,  $\epsilon = \delta/2^{j-p+2}$  and  $\tau = \delta/2^{j-p}$ . We have

$$\|[a, b] - Id\|_{\kappa_j - \delta/2^{j-p-1}} \leq \frac{2^{j+1-p}}{\delta} \frac{\delta}{2^{j+2}} \frac{\delta}{2^{j+2-p}} = \frac{\delta}{2^{j+3}}$$

and an analogous estimate holds for  $[b, a]$ . Since  $\kappa_{j+1} = \kappa_j - \delta/2^{j-p-1}$  we are done.  $\square$

*Proof of Proposition 3.1.* Since  $g(\mathbb{B}_{1-\delta/4}^n) \subset \mathbb{B}_1^n$  for any  $g \in \mathcal{S}$ , the pseudogroup generated by  $\{g|_{\mathbb{B}_{1-\delta/4}^n} : g \in \mathcal{S}\}$  is induced by  $\langle \mathcal{S} \rangle$  on  $\mathbb{B}_1^n$ .

Since  $(2p+4)\delta < 1/2$ , Lemmas 3.1 and 3.2 imply  $\|f - Id\|_{\frac{1}{2}} \leq \delta/2^{j+2}$  for all  $j \geq 0$  and  $f \in \mathcal{S}(j)$ .  $\square$

The next propositions are consequences of Proposition 3.1. The idea is that the non- $p$ -pseudo-solvability of a group of diffeomorphisms  $G$  can be used to show that the induced pseudogroup is non-discrete.

**Proposition 3.2.** *Fix  $p \in \mathbb{Z}_{\geq 0}$ ,  $\delta > 0$  with  $(p+2)\delta < 1/4$ . Let  $\mathcal{S}$  be a finite generator set of a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$ . Suppose  $G$  is non- $p$ -pseudo-solvable for  $\mathcal{S}$ . Furthermore suppose  $\|D_0\phi - Id\|_1 \leq \delta/8$  for any  $\phi \in \mathcal{S}$ . Then given any  $r > 0$  small enough there exists a sequence  $(f_m)_{m \geq 1}$  in the pseudogroup  $\mathcal{P}$  generated by  $\{f|_{\mathbb{B}_{r(1-\delta/4)}^n} : f \in \mathcal{S}\}$  on  $\mathbb{B}_r^n$  such that  $f_m$  is defined in  $\mathbb{B}_{r/2}^n$ ,  $(f_m)|_{\mathbb{B}_{r/2}^n} \not\equiv Id$  for any  $m \in \mathbb{N}$  and  $\lim_{m \rightarrow \infty} \|f_m - Id\|_{r/2} = 0$ . In particular all the points in  $\mathbb{B}_{r/2}^n$  except at most a countable union of proper analytic sets are recurrent for the action of  $\mathcal{P}$ .*

*Proof.* Let  $v_r(x_1, \dots, x_n) = (rx_1, \dots, rx_n)$  be the homothety of ratio  $r > 0$ . We define  $\phi_r = v_r^{-1} \circ \phi \circ v_r$  for  $\phi \in \mathcal{S}$  and  $r > 0$ .

Let  $\phi \in \mathcal{S}$ . The map  $(\phi_r)|_{\mathbb{B}_1^n}$  is injective and  $\phi_r - D_0\phi$  is bounded for  $r > 0$  small enough. The family  $(\phi_r)_{r>0}$  satisfies  $\lim_{r \rightarrow 0} \|\phi_r - D_0\phi\|_1 = 0$ . We consider  $r > 0$  small enough such that  $\|\phi_r - D_0\phi\|_1 \leq \delta/8$  for any  $\phi \in \mathcal{S}$ . We denote  $\mathcal{S}' = \{\phi_r : \phi \in \mathcal{S}\}$ .

Since  $G$  is not  $p$ -pseudo-solvable for  $\mathcal{S}$ , Proposition 3.1 implies the existence of a sequence  $f'_j : \mathbb{B}_{1/2}^n \rightarrow \mathbb{C}$  in the pseudogroup  $\mathcal{P}'$  generated by  $\{g|_{\mathbb{B}_{1-\delta/4}^n} : g \in \mathcal{S}'\}$  such that  $f'_j \in \mathcal{S}'_p(j)$  and  $0 < \|f'_j - Id\|_{1/2} \leq \delta/2^{j+2}$  for any  $j \geq 0$ . We denote  $f_j = v_r \circ f'_j \circ v_r^{-1}$  for  $j \geq 0$ . Then  $f_j$  is in the pseudogroup  $\mathcal{P}$ , is defined in  $\mathbb{B}_{r/2}^n$  and holds  $0 < \|f_j - Id\|_{r/2} \leq r\delta/2^{j+2}$  for any  $j \geq 0$ . Moreover since  $\mathcal{P}'$  is a pseudogroup on  $\mathbb{B}_1^n$  by Proposition 3.1, hence  $\mathcal{P}$  is a pseudogroup on  $\mathbb{B}_r^n$ .

We define  $T_j$  as the set of fixed points of  $f_j$  in  $\mathbb{B}_{r/2}^n$ . We denote  $S_j = \cap_{k \geq j} T_k$ . A sufficient condition guaranteeing the recurrence is  $(x_1, \dots, x_n) \notin \cup_{j \geq 0} S_j$ . Since  $S_j$  is an analytic set for  $j \geq 0$ , the set  $\cup_{j \geq 0} S_j$  is a countable union of proper analytic sets.  $\square$

**Proposition 3.3.** *Fix  $p \in \mathbb{Z}_{\geq 0}$ ,  $\delta > 0$  with  $(p+2)\delta < 1/4$ . Let  $G \subset \text{Diff}_u(\mathbb{C}^n, 0)$  be a non- $p$ -pseudo-solvable group. Then given any  $r > 0$  small enough there exists a sequence  $(f_j)_{j \geq 1}$  in the pseudogroup  $\mathcal{P}$  generated by  $\{f|_{\mathbb{B}_{r(1-\delta/4)}^n} : f \in G\}$  on  $\mathbb{B}_r^n$  such that  $f_j$  is defined in  $\mathbb{B}_{r/2}^n$ ,  $(f_j)|_{\mathbb{B}_{r/2}^n} \not\equiv Id$  for any  $j \geq 1$  and  $\lim_{j \rightarrow \infty} \|f_j - Id\|_{r/2} = 0$ . In*

particular all the points in  $\mathbb{B}_{r/2}^n$  except at most a countable union of proper analytic sets are recurrent for the action of  $\mathcal{P}$ .

We only consider diffeomorphisms  $f \in G$  well-defined, injective in  $\mathbb{B}_r^n$  and such that  $f(\mathbb{B}_{r(1-\delta/4)}^n) \subset \mathbb{B}_r^n$  as elements of  $\{f|_{\mathbb{B}_r^n} : f \in G\}$ .

*Proof.* Consider a finite generator set  $\mathcal{S}$  of  $G$ . Since  $j^1G$  (cf. Definition 2.5) consists of unipotent matrices, it is a group of upper triangular matrices up to a change of coordinates by Kolchin theorem (cf. [17][p. 35, Theorem 3\*]). In particular  $j^1G$  is nilpotent and then solvable. There exists  $m \in \mathbb{N}$  such that  $(j^1G)^{(m)} = \{Id\}$ . The set  $\mathcal{S}_p(j)$  is contained in  $G^{(l)}$  for any  $j \geq (l-1)p + l$ . We deduce that all elements of  $\mathcal{S}_p(j)$  are tangent to the identity for any  $j \geq (m-1)p + m$ . We define  $\mathcal{S}' = \mathcal{S}_p((m-1)p + m, mp + m)$ . We have  $\mathcal{S}_p(mp + m) \subset \mathcal{S}'_p(0)$  and a simple induction argument proves  $\mathcal{S}_p(mp + m + j) \subset \mathcal{S}'_p(j)$  for any  $j \in \mathbb{N}$ .

Since  $G$  is non- $p$ -pseudo-solvable for  $\mathcal{S}$ , the group  $\langle \mathcal{S}' \rangle$  is not  $p$ -pseudo-solvable for  $\mathcal{S}'$ . Since  $\langle \mathcal{S}' \rangle$  consists of tangent to the identity diffeomorphisms, we apply Proposition 3.2 to  $\langle \mathcal{S}' \rangle$  and  $\mathcal{S}'$ .  $\square$

It is natural that there is no condition on the generators being close to the  $Id$  in the previous proposition since it is implicit. Given a group  $G \subset \text{Diff}_u(\mathbb{C}^n, 0)$  generated by a finite set  $\mathcal{S}$  we can suppose that  $j^1G$  is upper triangular by Kolchin's theorem. Then we can suppose that  $j^1\mathcal{S}$  is contained in an arbitrary neighborhood of  $Id$  by making a linear change of coordinates. Thus the elements of  $\mathcal{S}$  can be considered arbitrarily close to the identity map.

#### 4. SUBGROUPS OF UNIPOTENT ELEMENTS

In this section we show that given a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  (or of its formal completion) of unipotent elements,  $G$  is solvable if and only if it is  $p$ -pseudo-solvable for some  $p \geq p(n)$  (Theorem 4.1).

**4.1. Formal diffeomorphisms and vector fields.** Let us recap some facts about pro-algebraic groups of formal diffeomorphisms in the next sections. This material can be found mostly in [14] and some points are expanded in [13].

We interpret elements of  $\text{Diff}(\mathbb{C}^n, 0)$  as operators acting on formal power series. Let  $\mathfrak{m}$  the maximal ideal of the ring  $\mathbb{C}[[x_1, \dots, x_n]]$  of formal power series. Any  $\phi \in \text{Diff}(\mathbb{C}^n, 0)$  induces an element  $\phi_k$  of  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$  by composition:

$$\begin{aligned} \phi_k &: \mathfrak{m}/\mathfrak{m}^{k+1} &\rightarrow \mathfrak{m}/\mathfrak{m}^{k+1} \\ g + \mathfrak{m}^{k+1} &\mapsto g \circ \phi + \mathfrak{m}^{k+1} \end{aligned}$$

Indeed  $\phi_k$  is an isomorphism of  $\mathbb{C}$ -algebras for any  $k \in \mathbb{N}$ . We consider the group  $D_k = \{\phi_k : \phi \in \text{Diff}(\mathbb{C}^n, 0)\}$ , it coincides with the subgroup of  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$  of isomorphisms of the  $\mathbb{C}$ -algebra  $\mathfrak{m}/\mathfrak{m}^{k+1}$ . In particular  $D_k$  is an algebraic subgroup of  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ .

**Definition 4.1.** We define the group of formal diffeomorphisms  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  as the projective limit  $\varprojlim_{k \in \mathbb{N}} D_k$ .

Given an element  $\phi = (A_k)_{k \geq 1}$  we have that  $(A_k(g + \mathfrak{m}^k))_{k \geq 1}$  converges in the  $\mathfrak{m}$ -adic topology, i.e. the Krull topology, to an element of  $\varprojlim_{k \in \mathbb{N}} \mathfrak{m}/\mathfrak{m}^{k+1}$  for any  $g \in \mathfrak{m}$ . Since  $\mathfrak{m}$  is equal to the inverse limit  $\varprojlim_{k \in \mathbb{N}} \mathfrak{m}/\mathfrak{m}^{k+1}$ , the limit  $\lim_{k \rightarrow \infty} A_k(g + \mathfrak{m}^k)$  exists in the Krull topology and belongs to  $\mathfrak{m}$ . We denote

$$\hat{\phi} = (\hat{\phi}_1, \dots, \hat{\phi}_n) := (\lim_{k \rightarrow \infty} A_k(x_1 + \mathfrak{m}^k), \dots, \lim_{k \rightarrow \infty} A_k(x_n + \mathfrak{m}^k)).$$

Since  $A_1$  is an isomorphism, the linear map  $j^1 \hat{\phi}$  is invertible. In this way we can interpret a formal diffeomorphism either as an element of  $\varprojlim_{k \in \mathbb{N}} D_k$  or as  $n$ -uple of elements of  $\mathfrak{m}$  whose first jet is invertible.

**Definition 4.2.** We define unipotent formal diffeomorphisms and groups analogously as in Definitions 2.6 and 2.7. We denote by  $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$  the subset of unipotent formal diffeomorphisms.

**Definition 4.3.** We define  $L_k$  as the Lie algebra of derivations of the  $\mathbb{C}$ -algebra  $\mathfrak{m}/\mathfrak{m}^{k+1}$ . We define the Lie algebra of (singular) formal vector fields  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$  as  $\varprojlim_{k \in \mathbb{N}} L_k$ .

Analogously as for formal diffeomorphisms, given  $(B_k)_{k \geq 1} \in \varprojlim L_k$  the limit  $\lim_{k \rightarrow \infty} B_k(g + \mathfrak{m}^k)$  is well-defined and belongs to  $\mathfrak{m}$  for any  $g \in \mathfrak{m}$ . We can identify  $(B_k)_{k \geq 1}$  with the expression

$$\lim_{k \rightarrow \infty} B_k(x_1 + \mathfrak{m}^k) \frac{\partial}{\partial x_1} + \dots + \lim_{k \rightarrow \infty} B_k(x_n + \mathfrak{m}^k) \frac{\partial}{\partial x_n}.$$

In this way it makes sense to consider the  $k$ -jet of a formal vector field.

**Definition 4.4.** We say that a formal vector field  $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$  is nilpotent if the first jet  $j^1 X$  (or  $D_0 X$ ) is a nilpotent linear transformation. We denote by  $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$  the subset of nilpotent formal vector fields.

*Remark 4.1.* It is a simple exercise to show that an element  $\phi = (A_k)_{k \geq 1} \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$  is unipotent if and only if  $A_k$  is unipotent for any  $k \in \mathbb{N}$ .

Analogously  $X = (B_k)_{k \geq 1} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$  is nilpotent if and only if  $B_k$  is nilpotent for any  $k \in \mathbb{N}$ .

*Remark 4.2.* The exponential establishes a bijection between nilpotent and unipotent matrices. Since  $L_k$  is the Lie algebra of  $D_k$ , the image of the nilpotent elements of  $L_k$  by the exponential map coincides with the set of unipotent elements of  $D_k$ . We remind the reader that the elements of  $L_k$  and  $D_k$  are interpreted as linear maps of  $\mathfrak{m}/\mathfrak{m}^{k+1}$ . In particular  $\exp : \hat{\mathfrak{X}}_N(\mathbb{C}^n, 0) \rightarrow \widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$  is a bijection.

**Definition 4.5.** Given  $\phi \in \widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$  we denote by  $\log \phi$  the unique element of  $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$  such that  $\phi = \exp(\log \phi)$ .

**Definition 4.6.** We define  $\hat{K}_n$  as the field of fractions of the local ring  $\mathbb{C}[[x_1, \dots, x_n]]$  of formal power series with complex coefficients.

**Definition 4.7.** Let  $\mathfrak{g}$  be a complex Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . We define  $\dim \mathfrak{g}$  as the dimension of the  $\hat{K}_n$ -vector space  $\mathfrak{g} \otimes_{\mathbb{C}} \hat{K}_n$ . We define  $\mathcal{M}(\mathfrak{g}) = \{g \in \hat{K}_n : X(g) \equiv 0 \ \forall X \in \mathfrak{g}\}$ , it is the field of formal meromorphic first integrals of  $\mathfrak{g}$ .

**4.2. Pro-algebraic groups.** Let us introduce the analogue of algebraic matrix groups for subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ .

**Definition 4.8.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . We define  $G_k = \overline{\{\phi_k : \phi \in G\}}^z$  where the Zariski-closure is considered in  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ . We define the pro-algebraic closure (also Zariski-closure)  $\overline{G}^z$  of  $G$  as the inverse limit  $\varprojlim_{k \in \mathbb{N}} G_k$ .

*Remark 4.3.* The group  $D_k$  is algebraic for  $k \in \mathbb{N}$ . As a consequence  $G_k$  is a subgroup of  $D_k$  for any  $k \in \mathbb{N}$  and then  $\overline{G}^z$  is a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0) = \varprojlim D_k$ . In particular the equality

$$\overline{G}^z = \{\phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : \phi_k \in G_k \ \forall k \in \mathbb{N}\}$$

holds.

*Remark 4.4.* The group  $\overline{G}^z$  is closed in the  $\mathfrak{m}$ -adic topology (the Krull topology) by construction.

**Definition 4.9.** We say that a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  is pro-algebraic if  $G = \overline{G}^z$ .

Notice that in [14] we use the notation  $\overline{G}^{(0)}$  instead of  $\overline{G}^z$ . Since in this paper we use three notions of closure we stress in the new notation that we are referring to the Zariski-closure.

Let us consider subgroups of formal diffeomorphisms contained in  $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ . The pro-algebraic theory is valid also for general subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  but we just apply it in this paper to the unipotent case. Let us introduce some basic properties of unipotent groups.



**Lemma 4.1.** *Let  $G$  be a unipotent subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . We have*

- $\overline{G}^z$  is contained in  $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ .
- The derived lengths of  $G$  and  $\overline{G}^z$  coincide. In particular  $G$  is solvable if and only if  $\overline{G}^z$  is solvable.
- The subset  $L(\overline{G}^z) := \{\log \phi : \phi \in \overline{G}^z\}$  is a Lie algebra of formal nilpotent vector fields. Moreover the map  $\exp : L(\overline{G}^z) \rightarrow \overline{G}^z$  is a bijection.
- The derived lengths of  $L(\overline{G}^z)$  and  $\overline{G}^z$  coincide. In particular  $G$  is solvable if and only if  $L(\overline{G}^z)$  is solvable.

*Proof.* All these results are proved in [14]. They correspond to Lemmas 4, 1 and Propositions 2, 3 respectively.  $\square$

**Definition 4.10.** We say that  $L(\overline{G}^z)$  is the Lie algebra of the pro-algebraic group  $\overline{G}^z$ .

*Remark 4.5.* Since  $\overline{G}^z$  is closed in the Krull topology, hence  $L(\overline{G}^z)$  is also closed in the Krull topology.

The algebraic properties of unipotent subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$  are codified in the Lie algebra of its pro-algebraic closure. The next result is Theorem 6 of [14], it displays part of the structure of a Lie algebra of formal vector fields.

**Proposition 4.1.** *Let  $\mathfrak{g}$  be a non-trivial complex Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . There exist ideals  $\mathfrak{h}, \mathfrak{j}$  of  $\mathfrak{g}$  such that*

- $\mathfrak{j} \subset \mathfrak{h}$  and  $\mathfrak{g}/\mathfrak{h}, \mathfrak{h}/\mathfrak{j}$  are abelian Lie algebras.
- $\dim \mathfrak{g} = \dim \mathfrak{h}$  and  $m := \dim \mathfrak{h} - \dim \mathfrak{j} > 0$ .
- $\mathfrak{g}/\mathfrak{h}$  is isomorphic to a complex Lie algebra of  $m \times m$  matrices with coefficients in  $\mathcal{M}(\mathfrak{g})$ .
- Let  $\{X_1, \dots, X_m\}$  be a base of  $(\mathfrak{g} \otimes_{\mathbb{C}} \hat{K}_n)/(\mathfrak{j} \otimes_{\mathbb{C}} \hat{K}_n)$  contained in  $\mathfrak{h}$ . Then given  $Z \in \mathfrak{h}$  there exists unique  $h_1, \dots, h_m \in \mathcal{M}(\mathfrak{g})$  such that  $Z - h_1 X_1 - \dots - h_m X_m \in \mathfrak{j} \otimes_{\mathbb{C}} \hat{K}_n$ .

*Remark 4.6.* Let us be a little more precise, the details can be found in the proof of Theorem 6 of [14]. The ideal  $\mathfrak{j}$  is equal to  $(\mathfrak{j} \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathfrak{g}$ . Moreover we have

$$\dim \mathfrak{j} = \dim \mathfrak{g}^{(a)}, \quad \mathfrak{j} \otimes_{\mathbb{C}} \hat{K}_n = \mathfrak{g}^{(a)} \otimes_{\mathbb{C}} \hat{K}_n \quad \text{and} \quad \mathfrak{j} = (\mathfrak{g}^{(a)} \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathfrak{g}$$

for the first  $a \in \{1, 2\}$  such that  $\dim \mathfrak{g}^{(a)} < \dim \mathfrak{g}$ . In particular  $\mathfrak{g}/\mathfrak{j}$  is abelian if and only if  $a = 1$ .

Given  $Z \in \mathfrak{g}$  we define

$$(4) \quad M(Z) = \begin{pmatrix} X_1(h_1) & X_1(h_2) & \dots & X_1(h_m) \\ X_2(h_1) & X_2(h_2) & \dots & X_2(h_m) \\ \vdots & \vdots & & \vdots \\ X_m(h_1) & X_m(h_2) & \dots & X_m(h_m) \end{pmatrix}.$$

where  $Z - h_1X_1 - \dots - h_mX_m \in \mathfrak{j} \otimes_{\mathbb{C}} \hat{K}_n$ . The map  $M : Z \mapsto M(Z)$  is a morphism of complex Lie algebras from  $\mathfrak{g}$  to the Lie algebra of  $m \times m$  matrices with coefficients in  $\mathcal{M}(\mathfrak{g})$ . The kernel of  $M$  is equal to  $\mathfrak{h}$ . Thus  $M : \mathfrak{g}/\mathfrak{h} \rightarrow M(\mathfrak{g})$  is an isomorphism of complex Lie algebras.

**4.3. Structure of a solvable Lie algebra.** In this section we improve Proposition 4.1 and provide a finer classification of Lie algebras of formal vector fields.

Let  $\tilde{\mathcal{L}}_1$  be a solvable Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . We denote  $\mathcal{M}_1 = \mathcal{M}(\tilde{\mathcal{L}}_1)$  and  $\mathcal{L}_1 = \tilde{\mathcal{L}}_1 \otimes_{\mathbb{C}} \mathcal{M}_1$ . Then  $\mathcal{L}_1$  is a Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$  such that  $\ell(\mathcal{L}_1) = \ell(\tilde{\mathcal{L}}_1)$ . Notice that  $\mathcal{M}_1 = \mathcal{M}(\mathcal{L}_1)$ . Our goal is constructing a Lie subalgebra  $\mathcal{L}$  of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$  that contains  $\mathcal{L}_1$ , it has similar algebraic properties as  $\mathcal{L}_1$  and also a simpler structure. In particular we provide a unique decomposition for the elements of  $\mathcal{L}$ .

We denote  $\mathfrak{g} = \mathcal{L}_1$ . Suppose  $\mathcal{L}_1 \neq 0$ . Consider the notations provided by Proposition 4.1. We denote  $b_1 = m = \dim \mathfrak{g} - \dim \mathfrak{j}$ . Since  $M$  is  $\mathcal{M}_1$ -linear, the Lie algebra  $M(\mathcal{L}_1)$  is a  $\mathcal{M}_1$ -vector space of dimension  $e_1$  less or equal than  $b_1^2$ . There exist elements  $W_1^1, \dots, W_{e_1}^1$  in  $\mathcal{L}_1$  such that  $\{M_{W_1^1}, \dots, M_{W_{e_1}^1}\}$  is a basis of  $M(\mathcal{L}_1)$ . As a consequence given  $X \in \mathcal{L}_1$  there exist unique  $f_1^1, \dots, f_{e_1}^1 \in \mathcal{M}_1$  such that  $M(X - \sum_{j=1}^{e_1} f_j^1 W_j^1) = 0$ . By construction  $X - \sum_{j=1}^{e_1} f_j^1 W_j^1$  belongs to  $\mathcal{L}_1$  and since its image by  $M$  vanishes, it belongs to  $\mathfrak{h}$ . Hence there exist unique  $h_1^1, \dots, h_{b_1}^1 \in \mathcal{M}_1$  such that

$$X - \sum_{j=1}^{e_1} f_j^1 W_j^1 - \sum_{k=1}^{b_1} h_k^1 X_k^1$$

belongs to  $(\mathfrak{j} \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathfrak{g}$  or equivalently to  $\mathfrak{j}$ . We define  $\tilde{\mathcal{L}}_2 = \mathfrak{j}$ ,  $\mathcal{M}_2 = \mathcal{M}(\tilde{\mathcal{L}}_2)$  and  $\mathcal{L}_2 = \tilde{\mathcal{L}}_2 \otimes_{\mathbb{C}} \mathcal{M}_2$ .

**Definition 4.11.** We say that  $\{Z_1^1, \dots, Z_{c_1}^1\}$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_1$  if it is a subset of  $\mathcal{L}_1$  such that the classes of its elements determine a basis of the  $\mathcal{M}_1$ -vector space  $\mathcal{L}_1/\tilde{\mathcal{L}}_2$ .

*Remark 4.7.* We define  $c_1 = b_1 + e_1$  and  $Z_j^1 = W_j^1$ ,  $Z_{k+e_1}^1 = X_k^1$  for  $1 \leq j \leq e_1$  and  $1 \leq k \leq b_1$ . The set  $\{Z_1^1, \dots, Z_{c_1}^1\}$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_1$ .

In particular we proved

$$(5) \quad \dim_{\mathcal{M}_1} \mathcal{L}_1 / \tilde{\mathcal{L}}_2 = \dim_{\mathcal{M}_1} M(\mathcal{L}_1) + \dim \mathcal{L}_1 - \dim \mathcal{L}_2 \leq n^2 + n.$$

*Remark 4.8.* The space  $\mathcal{L}_1 / \tilde{\mathcal{L}}_2$  is a  $\mathcal{M}_1$ -vector space of dimension  $b_1 + e_1$  whereas  $(\mathcal{L}_1 \otimes_{\mathbb{C}} \hat{K}_n) / (\tilde{\mathcal{L}}_2 \otimes_{\mathbb{C}} \hat{K}_n)$  is a  $\hat{K}_n$ -vector space of dimension  $b_1$ . The different dimensional type is due to the choice of the base field.

The Lie algebra  $\tilde{\mathcal{L}}_2$  is solvable since  $\tilde{\mathcal{L}}_2 \subset \mathcal{L}_1$ . Moreover  $\mathcal{L}_2$  is a solvable Lie algebra since  $\ell(\tilde{\mathcal{L}}_2) = \ell(\mathcal{L}_2)$ . The property  $[\mathcal{L}_1, \tilde{\mathcal{L}}_2] \subset \tilde{\mathcal{L}}_2$  implies that the elements in  $\mathcal{L}_1$  preserve the first integrals of  $\tilde{\mathcal{L}}_2$ . More precisely, we have

$$(6) \quad 0 = [Z, X](g) = Z(X(g)) - X(Z(g)) = Z(X(g)).$$

for all  $X \in \mathcal{L}_1$ ,  $Z \in \tilde{\mathcal{L}}_2$  and  $g \in \mathcal{M}_2$ . We deduce  $X(\mathcal{M}_2) \subset \mathcal{M}_2$  for any  $X \in \mathcal{L}_1$ . In particular we obtain  $[\mathcal{L}_1, \mathcal{L}_2] \subset \mathcal{L}_2$ . Since  $\mathcal{L}_1^{(2)} \subset \mathcal{L}_2$ ,  $\mathcal{L}_1 + \mathcal{L}_2$  is a solvable Lie algebra such that  $(\mathcal{L}_1 + \mathcal{L}_2)^{(2)} \subset \mathcal{L}_2$ .

Since  $\mathcal{L}_2$  is solvable we can repeat the process to obtain a Lie algebra  $\tilde{\mathcal{L}}_3$  if  $\mathcal{L}_2 \neq 0$ . Then we define  $\mathcal{M}_3 = \mathcal{M}(\tilde{\mathcal{L}}_3)$  and  $\mathcal{L}_3 = \tilde{\mathcal{L}}_3 \otimes_{\mathbb{C}} \mathcal{M}_3$ . We obtain a  $\mathcal{M}'$ -basis  $\{Z_1^2, \dots, Z_{c_2}^2\}$  for  $\mathcal{L}_2$  analogously as for  $\mathcal{L}_1$ . Given any  $X \in \mathcal{L}_2$  there exist unique  $\gamma_1^2, \dots, \gamma_{c_2}^2 \in \mathcal{M}_2$  such that  $X - \sum_{j=1}^{c_2} \gamma_j^2 Z_j^2 \in \tilde{\mathcal{L}}_3$ . We continue to obtain Lie algebras

$$\mathcal{L}_1, \dots, \mathcal{L}_m, \mathcal{L}_{m+1} = 0, \mathcal{L}_{m+2} = 0, \dots$$

where  $\mathcal{L}_m \neq 0$ . Notice that such  $m$  exists since  $\dim \mathcal{L}_{j+1} < \dim \mathcal{L}_j$  if  $\mathcal{L}_j \neq 0$ .

**Definition 4.12.** We say that  $\mathcal{L} = \sum_{j=1}^{\infty} \mathcal{L}_j = \mathcal{L}_1 + \dots + \mathcal{L}_m$  is the *extension Lie algebra* associated to  $\tilde{\mathcal{L}}_1$ . We define  $\mathcal{L}_j = 0$  for  $j \geq 1$  and  $\mathcal{L} = 0$  if  $\tilde{\mathcal{L}}_1 = 0$ .

*Remark 4.9.* The sum  $\sum_{j=1}^{\infty} \mathcal{L}_j$  is not a direct sum unless  $\mathcal{L}_j = 0$  for  $j \geq 2$ . We did not show that  $\mathcal{L}$  is a Lie algebra yet, it will be done in Proposition 4.2.

**Definition 4.13.** We denote  $\mathcal{M}_j = \mathcal{M}(\mathcal{L}_j)$ .

By construction we obtain  $\mathcal{L}_j^{(2)} \subset \mathcal{L}_{j+1}$  and  $[\mathcal{L}_j, \mathcal{L}_{j+1}] \subset \mathcal{L}_{j+1}$  for any  $1 \leq j \leq m$ . Next we generalize the latter property.

**Lemma 4.2.** *Let  $1 \leq j < k \leq m + 1$ . We have  $[\mathcal{L}_j, \mathcal{L}_k] \subset \mathcal{L}_k$ .*

*Proof.* The result holds for  $k = j + 1$  by construction. Let us prove that if it holds for some  $k \geq j + 1$  so it does for  $k + 1$ . Since  $[\mathcal{L}_j, \mathcal{L}_k] \subset \mathcal{L}_k$  we obtain  $[\mathcal{L}_j, (\mathcal{L}_k)^{(1)}] \subset (\mathcal{L}_k)^{(1)}$  and then  $[\mathcal{L}_j, (\mathcal{L}_k)^{(2)}] \subset (\mathcal{L}_k)^{(2)}$  by

the Jacobi identity. We have  $\tilde{\mathcal{L}}_{k+1} \otimes_{\mathbb{C}} \hat{K}_n = (\mathcal{L}_k)^{(a)} \otimes_{\mathbb{C}} \hat{K}_n$  for some  $a \in \{1, 2\}$  by Remark 4.6. Since

$$[\mathcal{L}_j, (\mathcal{L}_k)^{(a)} \otimes_{\mathbb{C}} \hat{K}_n] \subset (\mathcal{L}_k)^{(a)} \otimes_{\mathbb{C}} \hat{K}_n$$

for  $a \in \{1, 2\}$ , we deduce  $[\mathcal{L}_j, \tilde{\mathcal{L}}_{k+1} \otimes_{\mathbb{C}} \hat{K}_n] \subset \tilde{\mathcal{L}}_{k+1} \otimes_{\mathbb{C}} \hat{K}_n$ . Notice that  $(\tilde{\mathcal{L}}_{k+1} \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathcal{L}_k = \tilde{\mathcal{L}}_{k+1}$  by Remark 4.6. Since  $[\mathcal{L}_j, \mathcal{L}_k] \subset \mathcal{L}_k$  we obtain  $[\mathcal{L}_j, \tilde{\mathcal{L}}_{k+1}] \subset \tilde{\mathcal{L}}_{k+1}$ . This property implies  $X(\mathcal{M}_{k+1}) \subset \mathcal{M}_{k+1}$  for any  $X \in \mathcal{L}_j$  (cf. Equation (6)). We deduce  $[\mathcal{L}_j, \mathcal{L}_{k+1}] \subset \mathcal{L}_{k+1}$ .  $\square$

Our next goal is showing that the extension  $\mathcal{L}$  of  $\tilde{\mathcal{L}}_1$  has a simple structure.

**Proposition 4.2.** *Let  $\mathcal{L}$  be the extension Lie algebra associated to a solvable complex Lie subalgebra  $\tilde{\mathcal{L}}_1$  of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . Then  $\mathcal{L}$  is a solvable Lie algebra.*

*Proof.* Since  $\mathcal{L}_j^{(2)} \subset \mathcal{L}_{j+1}$  and  $[\mathcal{L}_j, \mathcal{L}_k] \subset \mathcal{L}_k$  for all  $1 \leq j \leq m$  and  $j \leq k$ , we deduce that  $\mathcal{L}$  is a Lie algebra such that

$$(\mathcal{L}_l + \dots + \mathcal{L}_m)^{(2)} \subset \mathcal{L}_{l+1} + \dots + \mathcal{L}_m$$

for any  $1 \leq l \leq m$ . In particular  $\ell(\mathcal{L})$  is less or equal than  $2m$ .  $\square$

The previous construction provides a sequence  $(Z_k^j)_{1 \leq j \leq m, 1 \leq k \leq c_j}$ , where  $\{Z_1^j, \dots, Z_{c_j}^j\}$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_j$  for any  $1 \leq j \leq m$ . Let us see that such sequence is associated to a unique decomposition of the elements of the extension Lie algebra.

**Proposition 4.3.** *Let  $\mathcal{L}$  be the extension Lie algebra associated to a solvable Lie subalgebra  $\tilde{\mathcal{L}}_1$  of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . Given an element  $X$  of  $\mathcal{L}$ , it can be written uniquely in the form  $\sum_{j=1}^m \sum_{k=1}^{c_j} \gamma_k^j Z_k^j$  where  $\gamma_k^j \in \mathcal{M}_j$  for all  $1 \leq j \leq m$  and  $1 \leq k \leq c_j$ . Indeed the elements of  $\mathcal{L}$  are exactly the elements of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$  that can be written in the previous form.*

*Proof.* Any element of the form  $\sum_{j=1}^m \sum_{k=1}^{c_j} \gamma_k^j Z_k^j$  (where  $\gamma_k^j \in \mathcal{M}_j$  for all  $1 \leq j \leq m$  and  $1 \leq k \leq c_j$ ) belongs to  $\mathcal{L}$  since  $\sum_{k=1}^{c_j} \gamma_k^j Z_k^j \in \mathcal{L}_j$  for any  $1 \leq j \leq m$ .

We denote  $\mathcal{K}_l = \mathcal{L}_l + \dots + \mathcal{L}_m$ . Let  $1 \leq l \leq m$ . Let us prove that any element  $X$  of  $\mathcal{K}_l$  can be written uniquely in the form  $\sum_{j=l}^m \sum_{k=1}^{c_j} \gamma_k^j Z_k^j$  where  $\gamma_k^j \in \mathcal{M}_j$  for all  $l \leq j \leq m$  and  $1 \leq k \leq c_j$ . The result is obvious for  $l = m$  by construction. Let us show that if the result holds for  $2 \leq l+1 \leq m$  then so it does for  $l$ . Consider  $X \in \mathcal{K}_l$ . It is of the

form  $\sum_{j=l}^m X_j$  where  $X_j \in \mathcal{L}_j$  for any  $l \leq j \leq m$ . If the decomposition  $X = \sum_{j=l}^m \sum_{k=1}^{c_j} \gamma_k^j Z_k^j$  exists then  $\sum_{k=1}^{c_l} \gamma_k^l Z_k^l$  is a formal vector field with  $\gamma_k^l \in \mathcal{M}_l$  for  $1 \leq k \leq c_l$  such that  $X - \sum_{k=1}^{c_l} \gamma_k^l Z_k^l \in \mathcal{K}_{l+1}$ . We use the previous condition to define  $\sum_{k=1}^{c_l} \gamma_k^l Z_k^l$ . Indeed since the condition is equivalent to  $X_l - \sum_{k=1}^{c_l} \gamma_k^l Z_k^l \in \tilde{\mathcal{L}}_{l+1}$ , we deduce that  $\sum_{k=1}^{c_l} \gamma_k^l Z_k^l$  is unique with the required properties. The result is a consequence of applying the induction hypothesis to  $X - \sum_{k=1}^{c_l} \gamma_k^l Z_k^l$ .  $\square$

**Definition 4.14.** We say that  $\mathcal{B} := \{Z_1^1, \dots, Z_{c_1}^1, \dots, Z_1^m, \dots, Z_{c_m}^m\}$  is a  $\mathcal{M}$ -basis of  $\mathcal{L} = \mathcal{L}_1 + \dots + \mathcal{L}_m$  if  $\{Z_1^j, \dots, Z_{c_j}^j\}$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_j$  for any  $1 \leq j \leq m$ . Let  $X \in \mathcal{L}$ ; the unique expression provided by Proposition 4.3 is called a  $\mathcal{M}$ -decomposition of  $X$  with respect to  $\mathcal{L}$  and  $\mathcal{B}$ .

Resuming we replace a solvable Lie subalgebra  $\tilde{\mathcal{L}}_1$  of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$  with a solvable Lie algebra  $\mathcal{L}$  that has a much simpler structure since its elements admit the unique expression provided by a  $\mathcal{M}$ -decomposition. Let us remark that a vector field or a diffeomorphism that normalizes  $\tilde{\mathcal{L}}_1$  also normalizes  $\mathcal{L}$ , making the extension Lie algebra suitable for the study of algebraic and geometrical problems.

**4.4. Increasing sequences of subgroups of a solvable group.** Let  $\tilde{\mathcal{L}}_1^0 \subset \tilde{\mathcal{L}}_1^1 \subset \dots \subset \tilde{\mathcal{L}}_1^j \subset \dots$  be an increasing sequence of solvable Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . We can associate Lie algebras  $\tilde{\mathcal{L}}_k^j$ ,  $\mathcal{L}_k^j$  and  $\mathcal{L}^j = \mathcal{L}_1^j + \dots + \mathcal{L}_{m_j}^j$  to  $\tilde{\mathcal{L}}_1^j$  as described in the previous section. We place the superindex  $j$  to all the objects associated to  $\tilde{\mathcal{L}}_1^j$ .

*Remark 4.10.* We will apply the results in this section to sequences  $J(0) \subset J(1) \subset \dots$  of subgroups of a solvable group  $\Gamma$  contained in  $\text{Diff}_u(\mathbb{C}^n, 0)$ . We define  $\tilde{\mathcal{L}}_1^j = L(\overline{J(j)}^z)$ . Since  $J(j) \subset J(j+1)$  for  $j \geq 0$ , we obtain  $\overline{J(j)}^z \subset \overline{J(j+1)}^z$  and

$$\tilde{\mathcal{L}}_1^j = L(\overline{J(j)}^z) \subset L(\overline{J(j+1)}^z) = \tilde{\mathcal{L}}_1^{j+1}$$

for  $j \geq 0$ . In particular we obtain an increasing sequence  $\tilde{\mathcal{L}}_1^0 \subset \tilde{\mathcal{L}}_1^1 \subset \dots$  of solvable Lie algebras (cf. Lemma 4.1).

We will associate to  $\mathcal{L}^j$  a list of invariants. We will see that the sequence of lists for  $j \geq 0$  is increasing in the lexicographical order. The equality of the invariants associated to  $\mathcal{L}^j$  and  $\mathcal{L}^{j+1}$  implies  $\mathcal{L}^j = \mathcal{L}^{j+1}$ . We will obtain  $\mathcal{L}^j = \mathcal{L}^{j+1}$  for some  $j \geq 0$  by showing that the list takes finitely many values.

**Lemma 4.3.** *There exists  $0 \leq j \leq n$  such that  $\dim \mathcal{L}_1^j = \dim \mathcal{L}_1^{j+1}$ . Moreover there exists  $0 \leq k < (n+1)^3$  such that  $\dim \mathcal{L}_1^k = \dim \mathcal{L}_1^{k+1}$ ,  $\dim(\mathcal{L}_1^k)^{(1)} = \dim(\mathcal{L}_1^{k+1})^{(1)}$  and  $\dim(\mathcal{L}_1^k)^{(2)} = \dim(\mathcal{L}_1^{k+1})^{(2)}$ .*

*Proof.* Since  $\dim \mathcal{L}_1^j = \dim \tilde{\mathcal{L}}_1^j$  for  $j \geq 1$ , we obtain an increasing sequence

$$0 \leq \dim \mathcal{L}_1^1 \leq \dim \mathcal{L}_1^2 \leq \dots \leq \dim \mathcal{L}_1^j \leq \dim \mathcal{L}_1^{j+1} \leq \dots \leq n.$$

Clearly there exists  $0 \leq j \leq n$  such that  $\dim \mathcal{L}_1^j = \dim \mathcal{L}_1^{j+1}$ .

Notice that  $(\tilde{\mathcal{L}}_1^j)^{(k)} \subset (\tilde{\mathcal{L}}_1^{j+1})^{(k)}$  and  $\dim(\mathcal{L}_1^j)^{(k)} = \dim(\tilde{\mathcal{L}}_1^j)^{(k)}$  for all  $j \geq 0$  and  $k \geq 0$ . As a consequence  $(\dim \mathcal{L}_1^j, \dim(\mathcal{L}_1^j)^{(1)}, \dim(\mathcal{L}_1^j)^{(2)})$  is increasing in every component and in particular for the lexicographical order. Since there are at most  $(n+1)^3$  3-uples, they coincide for some  $0 \leq k < (n+1)^3$ .  $\square$

**Definition 4.15.** We define  $I(j, k) = (0, 0, 0, 0)$  if  $\mathcal{L}_k^j = 0$  and

$$I(j, k) = (\dim(\mathcal{L}_k^j), \dim(\mathcal{L}_k^j)^{(1)}, \dim(\mathcal{L}_k^j)^{(2)}, \dim_{\mathcal{M}_k^j} \mathcal{L}_k^j / \tilde{\mathcal{L}}_{k+1}^j)$$

otherwise. We define the sequence  $I(j) = (I(j, 1), \dots, I(j, n))$ .

Let us study the behavior of the sequence  $(I(j, 1))_{j \geq 0}$  and its impact on the relation between  $\mathcal{L}_2^j$  and  $\mathcal{L}_2^{j+1}$ .

**Lemma 4.4.** *Let  $j \geq 0$  such that  $\dim(\mathcal{L}_1^j)^{(k)} = \dim(\mathcal{L}_1^{j+1})^{(k)}$  for any  $k \in \{0, 1, 2\}$ . Then*

- $\mathcal{L}_1^j \subset \mathcal{L}_1^{j+1}$ ,  $\tilde{\mathcal{L}}_2^j \subset \tilde{\mathcal{L}}_2^{j+1}$ ,  $\mathcal{L}_2^j \subset \mathcal{L}_2^{j+1}$  and  $\dim \mathcal{L}_2^j = \dim \mathcal{L}_2^{j+1}$ .
- $\mathcal{M}_1^j = \mathcal{M}_1^{j+1}$  and  $\mathcal{M}_2^j = \mathcal{M}_2^{j+1}$ .
- $I(j, 1) \leq I(j+1, 1)$ .

*Proof.* We deduce  $\tilde{\mathcal{L}}_1^j \otimes_{\mathbb{C}} \hat{K}_n = \tilde{\mathcal{L}}_1^{j+1} \otimes_{\mathbb{C}} \hat{K}_n$  since  $\tilde{\mathcal{L}}_1^j \subset \tilde{\mathcal{L}}_1^{j+1}$  and  $\dim \tilde{\mathcal{L}}_1^j = \dim \tilde{\mathcal{L}}_1^{j+1}$ . In particular we obtain  $\mathcal{M}_1^j = \mathcal{M}_1^{j+1}$ . Thus  $\mathcal{L}_1^j$  is contained in  $\mathcal{L}_1^{j+1}$ . Moreover  $(\mathcal{L}_1^j)^{(k)} \subset (\mathcal{L}_1^{j+1})^{(k)}$  for any  $k \in \{0, 1, 2\}$ .

We have  $\dim \tilde{\mathcal{L}}_2^l = \dim(\mathcal{L}_1^l)^{(k_l)}$  and  $\tilde{\mathcal{L}}_2^l = ((\mathcal{L}_1^l)^{(k_l)} \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathcal{L}_1^l$  for the first  $k_l \in \{1, 2\}$  such that  $\dim(\mathcal{L}_1^l)^{(k_l)} < \dim \mathcal{L}_1^l$  for  $l \geq 0$  by Remark 4.6. Let  $r := k_j = k_{j+1}$ ; we obtain  $\tilde{\mathcal{L}}_2^j \subset \tilde{\mathcal{L}}_2^{j+1}$  and

$$\dim \mathcal{L}_2^j = \dim \tilde{\mathcal{L}}_2^j = \dim(\mathcal{L}_1^j)^{(r)} = \dim(\mathcal{L}_1^{j+1})^{(r)} = \dim \tilde{\mathcal{L}}_2^{j+1} = \dim \mathcal{L}_2^{j+1}.$$

The properties  $\tilde{\mathcal{L}}_2^j \subset \tilde{\mathcal{L}}_2^{j+1}$  and  $\dim \mathcal{L}_2^j = \dim \mathcal{L}_2^{j+1}$  imply  $\mathcal{M}_2^j = \mathcal{M}_2^{j+1}$ . Hence we get  $\mathcal{L}_2^j \subset \mathcal{L}_2^{j+1}$ .

The natural map  $\varpi_j : \mathcal{L}_1^j / \tilde{\mathcal{L}}_2^j \rightarrow \mathcal{L}_1^{j+1} / \tilde{\mathcal{L}}_2^{j+1}$  is well-defined and  $\mathcal{M}_1$ -linear where  $\mathcal{M}_1 := \mathcal{M}_1^j = \mathcal{M}_1^{j+1}$ . Since

$$\tilde{\mathcal{L}}_2^j \otimes_{\mathbb{C}} \hat{K}_n = \tilde{\mathcal{L}}_2^{j+1} \otimes_{\mathbb{C}} \hat{K}_n, \quad \tilde{\mathcal{L}}_2^j = (\tilde{\mathcal{L}}_2^j \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathcal{L}_1^j$$

and  $\tilde{\mathcal{L}}_2^{j+1} = (\tilde{\mathcal{L}}_2^{j+1} \otimes_{\mathbb{C}} \hat{K}_n) \cap \mathcal{L}_1^{j+1}$ , the map  $\varpi_j$  is injective and we obtain  $\dim_{\mathcal{M}_1} \mathcal{L}_1^j / \tilde{\mathcal{L}}_2^j \leq \dim_{\mathcal{M}_1} \mathcal{L}_1^{j+1} / \tilde{\mathcal{L}}_2^{j+1}$ .  $\square$

**Corollary 4.1.** *The sequence  $(I(j, 1))_{j \geq 0}$  is increasing in the lexicographical order. Moreover  $I(j, 1) = I(j+1, 1)$  implies  $\mathcal{L}_2^j \subset \mathcal{L}_2^{j+1}$ .*

We define  $C = (n+1)^3(n^2+1)$ . We obtain

**Lemma 4.5.** *Suppose  $I(l, 1) = I(l+1, 1)$ . Then we have*

- $\mathcal{L}_1^l \subset \mathcal{L}_1^{l+1}$  and  $\dim(\mathcal{L}_1^l)^{(k)} = \dim(\mathcal{L}_1^{l+1})^{(k)}$  for any  $k \in \{0, 1, 2\}$ .
- $\dim \mathcal{L}_2^l = \dim \mathcal{L}_2^{l+1}$ ,  $\tilde{\mathcal{L}}_2^l \subset \tilde{\mathcal{L}}_2^{l+1}$  and  $\mathcal{L}_2^l \subset \mathcal{L}_2^{l+1}$ .
- $\mathcal{M}_1^l = \mathcal{M}_1^{l+1}$  and  $\mathcal{M}_2^l = \mathcal{M}_2^{l+1}$ .
- Any  $\mathcal{M}'$ -basis of  $\mathcal{L}_1^l$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_1^{l+1}$ .

Moreover, there exists  $0 \leq l < C$  such that  $I(l, 1) = I(l+1, 1)$ .

*Proof.* All the items except the last one are consequence of Lemma 4.4. Since  $\varpi_l : \mathcal{L}_1^l / \tilde{\mathcal{L}}_2^l \rightarrow \mathcal{L}_1^{l+1} / \tilde{\mathcal{L}}_2^{l+1}$  is injective by the proof of Lemma 4.4,  $\dim_{\mathcal{M}_1^l} \mathcal{L}_1^l / \tilde{\mathcal{L}}_2^l = \dim_{\mathcal{M}_1^{l+1}} \mathcal{L}_1^{l+1} / \tilde{\mathcal{L}}_2^{l+1}$  and  $\mathcal{M}_1^l = \mathcal{M}_1^{l+1}$ , the map  $\varpi_l$  is an isomorphism. Hence a  $\mathcal{M}'$ -basis of  $\mathcal{L}_1^l$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_1^{l+1}$ .

Suppose that the first three coordinates of  $I(q, 1)$  does not change for  $b \leq q \leq c$ . The sequence  $(\dim_{\mathcal{M}_1} \mathcal{L}_1^j / \tilde{\mathcal{L}}_2^j)_{b \leq j \leq c}$  is increasing by Lemma 4.4 where  $\mathcal{M}_1 = \mathcal{M}_1^j$  for  $b \leq j \leq c$ . Since

$$0 \leq \dim_{\mathcal{M}_1} \mathcal{L}_1^k / \tilde{\mathcal{L}}_2^k - \dim_{\mathcal{M}_1} \mathcal{L}_1^j / \tilde{\mathcal{L}}_2^j \leq n^2$$

for all  $b \leq j \leq k \leq c$  by Equation (5), the sequence  $(\dim_{\mathcal{M}_1} \mathcal{L}_1^j / \tilde{\mathcal{L}}_2^j)_{b \leq j \leq c}$  takes at most  $n^2 + 1$  values. Hence the sequence  $(I(j, 1))_{j \geq 0}$  takes at most  $C$  values. In particular there exists  $0 \leq l < C$  such that  $I(l, 1) = I(l+1, 1)$ .  $\square$

By applying Lemma 4.5 at most  $n$  times we obtain

**Proposition 4.4.** *Let  $\tilde{\mathcal{L}}_1^0 \subset \tilde{\mathcal{L}}_1^1 \subset \dots$  be an increasing sequence of solvable Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0) \otimes_{\mathbb{C}} \hat{K}_n$ . The sequence  $(I(j))_{j \geq 0}$  is increasing in the lexicographical order. Moreover there exists  $0 \leq q < C^n$  such that  $I(q) = I(q+1)$ . In particular we obtain*

- $\mathcal{L}^q = \mathcal{L}_1^q + \dots + \mathcal{L}_m^q$  and  $\mathcal{L}^{q+1} = \mathcal{L}_1^{q+1} + \dots + \mathcal{L}_m^{q+1}$ .
- $\dim(\mathcal{L}_j^q)^{(k)} = \dim(\mathcal{L}_j^{q+1})^{(k)}$  for all  $1 \leq j \leq m$  and  $k \in \{0, 1, 2\}$ .
- $\tilde{\mathcal{L}}_j^q \subset \tilde{\mathcal{L}}_j^{q+1}$ ,  $\mathcal{L}_j^q \subset \mathcal{L}_j^{q+1}$  and  $\mathcal{M}_j^q = \mathcal{M}_j^{q+1}$  for any  $1 \leq j \leq m$ .
- Any  $\mathcal{M}'$ -basis of  $\mathcal{L}_j^q$  is a  $\mathcal{M}'$ -basis of  $\mathcal{L}_j^{q+1}$  for any  $1 \leq j \leq m$ .
- Any  $\mathcal{M}$ -basis of  $\mathcal{L}^q$  is a  $\mathcal{M}$ -basis of  $\mathcal{L}^{q+1}$ .
- $\mathcal{L}^q = \mathcal{L}^{q+1}$ .



*Proof.* Corollary 4.1 implies that we can apply iteratively Lemma 4.5 to obtain the increasing nature of  $(I(j))_{j \geq 0}$  and the existence of  $q$  satisfying all properties but the last one. Let us prove  $\mathcal{L}^q = \mathcal{L}^{q+1}$ . We choose a  $\mathcal{M}$ -basis  $\mathcal{B}$  of  $\mathcal{L}^q$ . It is also a  $\mathcal{M}$ -basis of  $\mathcal{L}^{q+1}$ . Since  $\mathcal{M}_j^q = \mathcal{M}_j^{q+1}$  for any  $1 \leq j \leq m$ , the set of  $\mathcal{M}$ -decompositions with respect to  $\mathcal{L}^q$  and  $\mathcal{B}$  coincides with the set of  $\mathcal{M}$ -decompositions with respect to  $\mathcal{L}^{q+1}$  and  $\mathcal{B}$ . Proposition 4.3 implies  $\mathcal{L}^q = \mathcal{L}^{q+1}$ .  $\square$

*Remark 4.11.* Notice that if  $I(j, k) = I(j+1, k)$  then the first coordinates of  $I(j, k+1)$  and  $I(j+1, k+1)$  coincide by Lemma 4.5. As a consequence we can replace  $C^n$  in Proposition 4.4 with  $n(n^2(n^2+1))^n$  and the result still holds.

**4.5. Solvability of  $p$ -pseudo-solvable subgroups.** This section is devoted to prove the following theorem:

**Theorem 4.1.** *Let  $G \subset \text{Diff}_u(\mathbb{C}^n, 0)$  be a finitely generated group and  $p \geq C^n$ . Suppose that  $G$  is  $p$ -pseudo-solvable for some finite generator set  $\mathcal{S}$ . Then  $G$  is solvable.*

Let  $m_0 \geq 0$  be the first index such that  $\mathcal{S}(m_0) = \{Id\}$ . We define

$$\mathcal{S}(j, k) = \mathcal{S}(j) \cup \mathcal{S}(j+1) \cup \dots \cup \mathcal{S}(k),$$

$G(j, k) = \langle \mathcal{S}(j, k) \rangle$  and  $\Gamma(l) = G(l, m_0)$  for  $0 \leq j \leq k$  and  $0 \leq l \leq m_0$ . Our goal is proving that  $\Gamma(0)$  is solvable. It is obvious that  $\Gamma(m_0)$  is solvable since it is the trivial group. We will show that whenever  $\Gamma(m+1)$  is solvable for  $0 \leq m \leq m_0 - 1$  then  $\Gamma(m)$  is solvable.

Let  $\mathcal{L}^j$  be the extension Lie algebra associated to  $G(m+1, j)$  (where  $\tilde{\mathcal{L}}_1^j := L(\overline{G(m+1, j)})^z$ ) for  $j \geq m+1$ . Since  $(G(m+1, j))_{j \geq m+1}$  is an increasing sequence of subgroups of the solvable group  $\overline{\Gamma(m+1)}^z$  (cf. Lemma 4.1), there exists  $m+1 \leq q \leq m+p$  such that  $\mathcal{L}^q = \mathcal{L}^{q+1}$  by Remark 4.10 and Proposition 4.4. Moreover we have  $\mathcal{L}^q = \mathcal{L}_1^q + \dots + \mathcal{L}_m^q$ ,  $\mathcal{L}^{q+1} = \mathcal{L}_1^{q+1} + \dots + \mathcal{L}_m^{q+1}$  and  $\mathcal{L}^q$  and  $\mathcal{L}^{q+1}$  satisfy all conditions in Proposition 4.4. We denote  $\mathcal{L} = \mathcal{L}^q$  and  $\mathcal{M}_j = \mathcal{M}_j^q$ .

The idea is extending  $\varphi G(m+1, q) \varphi^{-1} \subset G(m+1, q+1)$  for  $\varphi \in \mathcal{S}(q-p, q)$  to the extension Lie algebras  $\mathcal{L}^q$  and  $\mathcal{L}^{q+1}$ . This is natural since we enlarge the set of coefficients by adding first integrals of Lie subalgebras canonically associated to the initial Lie algebra.

**Lemma 4.6.** *We have  $\mathcal{M}_j \circ \varphi = \mathcal{M}_j$ ,  $\varphi_* \tilde{\mathcal{L}}_j^q \subset \tilde{\mathcal{L}}_j^{q+1}$  and  $\varphi_* \mathcal{L}_j^q \subset \mathcal{L}_j^{q+1}$  for all  $1 \leq j \leq m$  and  $\varphi \in \mathcal{S}(q-p, q)$ . In particular  $\varphi_* \mathcal{L} = \mathcal{L}$  for any  $\varphi \in \mathcal{S}(q-p, q)$ .*

*Proof.* We have  $\varphi G(m+1, q) \varphi^{-1} \subset G(m+1, q+1)$  for any  $\varphi \in \mathcal{S}(q-p, q)$  by the definition of the sets  $\mathcal{S}(l)$  for  $l \geq 0$ . We obtain

$$\overline{\varphi G(m+1, q)}^z \varphi^{-1} \subset \overline{G(m+1, q+1)}^z$$

and  $\varphi_* L(\overline{G(m+1, q)}^z) \subset L(\overline{G(m+1, q+1)}^z)$  for any  $\varphi \in \mathcal{S}(q-p, q)$ . Equivalently we have  $\varphi_* \tilde{\mathcal{L}}_1^q \subset \tilde{\mathcal{L}}_1^{q+1}$  for any  $\varphi \in \mathcal{S}(q-p, q)$ .

Since  $\dim \tilde{\mathcal{L}}_1^q = \dim \tilde{\mathcal{L}}_1^{q+1}$ , we deduce  $\varphi_*(\tilde{\mathcal{L}}_1^q \otimes_{\mathbb{C}} \hat{K}_n) = \tilde{\mathcal{L}}_1^{q+1} \otimes_{\mathbb{C}} \hat{K}_n$ . In particular we obtain  $\mathcal{M}_1 \circ \varphi = \mathcal{M}_1$  for any  $\varphi \in \mathcal{S}(q-p, q)$ . The properties  $\varphi_* \tilde{\mathcal{L}}_1^q \subset \tilde{\mathcal{L}}_1^{q+1}$  and  $\varphi_* \mathcal{M}_1 = \mathcal{M}_1$  imply  $\varphi_* \mathcal{L}_1^q \subset \mathcal{L}_1^{q+1}$  for any  $\varphi \in \mathcal{S}(q-p, q)$ .

Fix  $\varphi \in \mathcal{S}(q-p, q)$ . Let us prove next that  $\varphi_* \mathcal{L}_j^q \subset \mathcal{L}_j^{q+1}$  implies  $\varphi_* \mathcal{M}_{j+1} = \mathcal{M}_{j+1}$ ,  $\varphi_* \tilde{\mathcal{L}}_{j+1}^q \subset \tilde{\mathcal{L}}_{j+1}^{q+1}$  and  $\varphi_* \mathcal{L}_{j+1}^q \subset \mathcal{L}_{j+1}^{q+1}$  for any  $1 \leq j < m$ . Since  $\varphi_* \mathcal{L}_j^q \subset \mathcal{L}_j^{q+1}$  we obtain  $\varphi_*(\mathcal{L}_j^q)^{(k)} \subset (\mathcal{L}_j^{q+1})^{(k)}$  for  $k \geq 0$ . Since  $\dim(\mathcal{L}_j^q)^{(k)} = \dim(\mathcal{L}_j^{q+1})^{(k)}$  for  $k \in \{0, 1, 2\}$ , the equality  $(\mathcal{L}_j^q)^{(k)} \otimes_{\mathbb{C}} \hat{K}_n = (\mathcal{L}_j^{q+1})^{(k)} \otimes_{\mathbb{C}} \hat{K}_n$  holds for  $k \in \{0, 1, 2\}$ . There exists  $k' \in \{1, 2\}$  such that

$$\tilde{\mathcal{L}}_{j+1}^l = \mathcal{L}_j^l \cap ((\mathcal{L}_j^l)^{(k')} \otimes_{\mathbb{C}} \hat{K}_n) \text{ and } \mathcal{M}_{j+1}^l = \mathcal{M}((\mathcal{L}_j^l)^{(k')})$$

for any  $l \in \{q, q+1\}$ . Since  $\varphi_*((\mathcal{L}_j^q)^{(k')} \otimes_{\mathbb{C}} \hat{K}_n) = (\mathcal{L}_j^{q+1})^{(k')} \otimes_{\mathbb{C}} \hat{K}_n$  we deduce  $\mathcal{M}_{j+1} \circ \varphi = \mathcal{M}_{j+1}$  and  $\varphi_* \tilde{\mathcal{L}}_{j+1}^q \subset \tilde{\mathcal{L}}_{j+1}^{q+1}$ . These properties lead to  $\varphi_* \mathcal{L}_{j+1}^q \subset \mathcal{L}_{j+1}^{q+1}$  for any  $\varphi \in \mathcal{S}(q-p, q)$ .

Given  $\varphi \in \mathcal{S}(q-p, q)$ , we have

$$\varphi_* \mathcal{L} = \varphi_*(\mathcal{L}_1^q + \dots + \mathcal{L}_m^q) \subset \mathcal{L}_1^{q+1} + \dots + \mathcal{L}_m^{q+1} = \mathcal{L}.$$

We remind that  $\varphi^{-1} \in \mathcal{S}(q-p, q)$  if  $\varphi \in \mathcal{S}(q-p, q)$ . Hence the properties  $\varphi_* \mathcal{L} \subset \mathcal{L}$  and  $(\varphi^{-1})_* \mathcal{L} \subset \mathcal{L}$  imply  $\varphi_* \mathcal{L} = \mathcal{L}$  for any  $\varphi \in \mathcal{S}(q-p, q)$ .  $\square$

Consider the Lie algebra  $\mathfrak{g}$  of the pro-algebraic group  $\overline{G}^z$ . We define  $\mathfrak{h} = \mathfrak{g} \cap \mathcal{L}$ . We denote by  $\overline{\mathfrak{h}}^k$  the closure of  $\mathfrak{h}$  in the Krull topology (it can be proved that  $\overline{\mathfrak{h}}^k$  is equal to  $\mathfrak{h}$  but this result will not be necessary in the following). Remark 4.5 implies that  $\overline{\mathfrak{h}}^k$  is a complex Lie subalgebra of  $\mathfrak{g}$ .

**Proposition 4.5.** *The set  $\exp(\overline{\mathfrak{h}}^k) := \{\exp(X) : X \in \overline{\mathfrak{h}}^k\}$  is a solvable pro-algebraic subgroup of  $\overline{G}^z$  with Lie algebra  $\overline{\mathfrak{h}}^k$ .*

*Proof.* The Lie correspondence is explicit in the unipotent case: the Lie algebras of unipotent pro-algebraic groups are the Lie algebras of formal nilpotent vector fields that are closed in the Krull topology [13]. Since  $\mathfrak{g}$  and then  $\overline{\mathfrak{h}}^k$  consist of nilpotent elements, we obtain that  $\overline{\mathfrak{h}}^k$

is the Lie algebra of a unipotent pro-algebraic group. Such a group is equal to  $\exp(\overline{\mathfrak{h}}^k)$  by Lemma 4.1.

Let us show that  $\overline{\mathfrak{h}}^k$  is solvable. Since  $(\overline{\mathfrak{l}}^k)^{(1)} \subset \overline{\mathfrak{l}^{(1)}}^k$  for any Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ , we obtain  $(\overline{\mathfrak{h}}^k)^{(j)} \subset \overline{\mathfrak{h}^{(j)}}^k$  for any  $j \geq 0$ . In particular we get  $\ell(\overline{\mathfrak{h}}^k) \leq \ell(\mathfrak{h}) \leq \ell(\mathcal{L}) \leq 2m$  by Proposition 4.2. Since  $\overline{\mathfrak{h}}^k$  is solvable, the group  $\exp(\overline{\mathfrak{h}}^k)$  is solvable by Lemma 4.1.  $\square$

**Lemma 4.7.**  $\exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$  is a normal subgroup of  $\Gamma(q-p)$  that contains  $\Gamma(m+1)$ .

*Proof.* Let  $\varphi \in \mathcal{S}(q-p, q)$ . We have  $\varphi_*\mathcal{L} = \mathcal{L}$  by Lemma 4.6. Moreover  $\varphi \in G$  implies  $\varphi \overline{G}^z \varphi^{-1} = \overline{G}^z$  and then  $\varphi_*\mathfrak{g} = \mathfrak{g}$ . We deduce  $\varphi_*\mathfrak{h} = \mathfrak{h}$  and then  $\varphi_*\overline{\mathfrak{h}}^k = \overline{\mathfrak{h}}^k$ .

The infinitesimal generator  $\log \eta$  belongs to  $\mathcal{L}$  and then to  $\overline{\mathfrak{h}}^k$  for any  $\eta \in \mathcal{S}(q)$ . In particular  $\eta$  belongs to  $\exp(\overline{\mathfrak{h}}^k)$ .

Fix  $j \geq q$ . We claim that that  $\eta \in \exp(\overline{\mathfrak{h}}^k)$  and  $\varphi_*\overline{\mathfrak{h}}^k = \overline{\mathfrak{h}}^k$  for all  $\eta \in \mathcal{S}(j)$  and  $\varphi \in \mathcal{S}(j-p, j)$ . The proof is by induction on  $j$ . We already proved the result for  $j = q$ . Let us show that if it holds for  $j$  then so it does for  $j+1$ . Fix  $\phi \in \mathcal{S}(j)$  and  $\varphi \in \mathcal{S}(j-p, j)$ . Since  $\log \phi \in \overline{\mathfrak{h}}^k$  and  $\varphi_*\overline{\mathfrak{h}}^k = \overline{\mathfrak{h}}^k$ , we deduce  $\varphi \circ \phi \circ \varphi^{-1} \in \exp(\overline{\mathfrak{h}}^k)$ . The commutators

$$[\varphi, \phi] = (\varphi \circ \phi \circ \varphi^{-1}) \circ \phi^{-1} \text{ and } [\phi, \varphi] = \phi \circ (\varphi \circ \phi^{-1} \circ \varphi^{-1})$$

are compositions of elements of  $\exp(\overline{\mathfrak{h}}^k)$  and then belong to  $\exp(\overline{\mathfrak{h}}^k)$  by Proposition 4.5. By varying  $\varphi \in \mathcal{S}(j-p, j)$  and  $\phi \in \mathcal{S}(j)$  we obtain that  $\mathcal{S}(j+1) \subset \exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$ . Moreover  $\eta_*\overline{\mathfrak{h}}^k$  is equal to  $\overline{\mathfrak{h}}^k$  for any  $\eta \in \mathcal{S}(j+1)$  since  $\eta \in \exp(\overline{\mathfrak{h}}^k)$ .

We proved  $\varphi_*\overline{\mathfrak{h}}^k = \overline{\mathfrak{h}}^k$  for any  $\varphi \in \cup_{j \geq q-p} \mathcal{S}(j)$ . Thus  $\varphi$  normalizes  $\overline{\mathfrak{h}}^k$  for any  $\varphi \in \langle \cup_{j \geq q-p} \mathcal{S}(j) \rangle$ . Since  $\Gamma(q-p) = \langle \cup_{j \geq q-p} \mathcal{S}(j) \rangle$ , we deduce that  $\varphi$  normalizes  $\exp(\overline{\mathfrak{h}}^k)$  for any  $\varphi \in \Gamma(q-p)$ . Hence  $\exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$  is a normal subgroup of  $\Gamma(q-p)$ .

By construction  $\mathcal{S}(m+1, q)$  is contained in  $\exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$ . We proved  $\cup_{j \geq q} \mathcal{S}(j) \subset \exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$ . Since  $\exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$  is a group, the group  $\Gamma(m+1) = \langle \cup_{j > m} \mathcal{S}(j) \rangle$  is contained in  $\exp(\overline{\mathfrak{h}}^k) \cap \Gamma(q-p)$ .  $\square$

The next proposition completes the proof of the inductive step and as a consequence the proof of Theorem 4.1.

**Proposition 4.6.**  $\Gamma(m)$  is solvable.

*Proof.* Since  $q - p \leq m$ , the group  $\Gamma(m) \cap \exp(\bar{\mathfrak{h}}^k)$  is normal in  $\Gamma(m)$  by Lemma 4.7. We define the group  $H = \Gamma(m)/(\Gamma(m) \cap \exp(\bar{\mathfrak{h}}^k))$ . The property  $\Gamma(m+1) \subset \exp(\bar{\mathfrak{h}}^k)$  (Lemma 4.7) implies that  $H$  is generated by the classes of elements of  $\mathcal{S}(m)$ . Since the commutator of elements of  $\mathcal{S}(m)$  belongs to  $\mathcal{S}(m+1)$ , the group  $H$  is abelian. The group  $\Gamma(m)^{(1)}$  is contained in  $\exp(\bar{\mathfrak{h}}^k)$  and hence  $\Gamma(m)$  is solvable by Proposition 4.5.  $\square$

**4.6. Consequences.** We show that, given a pseudogroup induced by a non-solvable group of unipotent local diffeomorphisms, all points are recurrent outside of a measure zero set. This result was proved in dimension 2 by Rebelo and Reis [12].

**Proposition 4.7.** *Let  $G \subset \text{Diff}_u(\mathbb{C}^n, 0)$  be a non-solvable group. Then there exist  $r > 0$  and a sequence  $(f_j)_{j \geq 1}$  in the pseudogroup  $\mathcal{P}$  generated by  $\{f|_{\mathbb{B}_r^n} : f \in G\}$  such that  $f_j$  is defined in  $\mathbb{B}_{r/2}^n$ ,  $(f_j)|_{\mathbb{B}_{r/2}^n} \neq \text{Id}$  for any  $j \geq 1$  and  $\lim_{j \rightarrow \infty} \|f_j - \text{Id}\|_{r/2} = 0$ . In particular all the points in  $\mathbb{B}_{r/2}^n$  except at most a countable union of proper analytic sets are recurrent for the action of  $\mathcal{P}$ .*

*Proof.* A unipotent subgroup  $H$  of  $\text{Diff}(\mathbb{C}^n, 0)$  is solvable if and only if  $\ell(H) \leq 2n - 1$  [14][Theorem 4]. We have  $G^{(2n-1)} = \cup H^{(2n-1)}$  where the union is considered over the finitely generated subgroups of  $G$ . Hence up to replace  $G$  with one of its subgroups we can suppose that  $G$  is finitely generated. Fix  $p \in \mathbb{N}$  such that Theorem 4.1 holds. Hence  $G$  is non- $p$ -pseudo-solvable by Theorem 4.1. The remainder of the proof is an immediate consequence of Proposition 3.3.  $\square$

## 5. LINEAR GROUPS

Let us deal with the cases in Theorem 1 besides the first one, that was already settled in Proposition 4.7. The other cases are of linear type, indeed we will construct free subgroups of  $j^1G$  with free generators arbitrarily close to  $\text{Id}$  by using the Tits alternative [19]. In this way we obtain free subgroups of  $G$ ; they are clearly non- $p$ -pseudo-solvable for any  $p \in \mathbb{N}$  (cf. Lemma 5.3) and hence we can apply Proposition 3.2 to obtain recurrent points.

The linear part  $j^1G$  of a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  satisfies the Tits alternative, i.e. either  $j^1G$  is virtually solvable or it contains a non-abelian free group. A more precise result by Breuillard and Gelfander is the topological Tits alternative: a subgroup of  $\text{GL}(n, \mathbb{C})$  either contains an open solvable subgroup or a non-abelian dense free subgroup [3]. We will use this kind of ideas in sections 5.1, 5.2 and 5.3 to obtain free

subgroups of linear groups. At that point we will apply these results to the study of groups of local diffeomorphisms to show Theorem 1.

**Definition 5.1.** Let  $H$  be a subgroup of  $\mathrm{GL}(n, \mathbb{C})$ . We denote by  $\overline{H}$  the topological closure of  $H$ . It is well-known that  $\overline{H}$  is a real Lie group (cf. [7][p. 52, Corollary 2.33]). We denote by  $\overline{H}_0$  its connected component of the identity.

**5.1. Groups without hyperbolic elements.** In this section we focus on Case (2) of Theorem 1.

**Definition 5.2.** We say that an element  $A$  of  $\mathrm{GL}(n, \mathbb{C})$  is hyperbolic if  $\mathrm{spec}(A) \not\subset \mathbb{S}^1$ . We say that  $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$  is hyperbolic if  $D_0\phi$  is hyperbolic.

The main result of this section is next theorem

**Theorem 5.1.** *Let  $H$  be a subgroup of  $\mathrm{GL}(n, \mathbb{C})$ . Suppose that  $H$  is non-virtually solvable and does not contain hyperbolic elements. Then given any neighborhood  $V$  of  $\mathrm{Id}$  in  $\mathrm{GL}(n, \mathbb{C})$  there exists  $A, B \in H \cap V$  such that  $A$  and  $B$  are free generators of the free group  $\langle A, B \rangle$ . In particular the group  $\overline{H}_0$  is non-virtually solvable.*

Let  $H$  be a group satisfying the hypotheses of Theorem 5.1. It is easier to prove Theorem 5.1 if  $H$  is irreducible since then we can use Burnside's theorem. Anyway we associate to  $H$  a sequence of irreducible representations that are useful to show Theorem 5.1.

Consider a sequence

$$(7) \quad \{0\} = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_r = \mathbb{C}^n$$

where  $V_j$  is  $H$ -invariant for any  $0 \leq j \leq r$ . An example is provided by  $r = 1$ ,  $V_0 = \{0\}$  and  $V_1 = \mathbb{C}^n$ . The group  $H$  acts on the vector space  $V_{j+1}/V_j$  for  $0 \leq j < r$ . If the action is not irreducible then there exists a  $H$ -invariant vector space  $V_{j+1/2}$  such that  $V_j \subsetneq V_{j+1/2} \subsetneq V_{j+1}$ , hence we can refine the sequence (7) by introducing the subspace  $V_{j+1/2}$ . Since we can not refine indefinitely, there exists a sequence (7) of  $H$ -invariant subspaces such that the action of  $H$  on  $V_{j+1}/V_j$  is irreducible for any  $0 \leq j \leq r$ . We denote  $d_j = \dim V_j$  and  $c_j = \dim(V_j/V_{j-1})$ . We define  $H_j$  as the group induced by  $H$  on  $V_j/V_{j-1}$  for any  $1 \leq j \leq r$ . The group  $H_j$  is irreducible by construction and it does not contain hyperbolic elements by hypothesis. Moreover we have

**Lemma 5.1.** *The group  $H_j$  is relatively compact in  $\mathrm{GL}(V_j/V_{j-1})$  for any  $1 \leq j \leq r$ .*

*Proof.* Since  $H_j$  is an irreducible subgroup of  $\mathrm{GL}(V_j/V_{j-1})$ , Burnside's theorem implies that there exist  $c_j^2$   $\mathbb{C}$ -linearly independent elements  $g_1, \dots, g_{c_j^2}$  of  $H_j$  (cf. [20][p. 11, Corollary 1.17]).

Consider now the symmetric bilinear form  $\alpha : M_{c_j}(\mathbb{C}) \times M_{c_j}(\mathbb{C}) \rightarrow \mathbb{C}$  defined by  $\alpha(f, h) = \mathrm{trace}(fh)$  where  $M_{c_j}(\mathbb{C})$  is the vector space of  $c_j \times c_j$  complex matrices. The bilinear form  $\alpha$  is non-degenerate; indeed given  $f \in M_{c_j}(\mathbb{C}) \setminus \{0\}$  there exists  $v \in \mathbb{C}^{c_j}$  such that  $f(v) \neq 0$ . Consider a base  $\{w_1, \dots, w_{c_j}\}$  of  $\mathbb{C}^{c_j}$  such that  $w_1 = f(v)$  and a linear map  $h : \mathbb{C}^{c_j} \rightarrow \mathbb{C}^{c_j}$  with  $h(w_1) = v$  and  $h(w_k) = 0$  for any  $2 \leq k \leq c_j$ . The map  $fh$  satisfies  $fh(w_1) = w_1$  and  $fh(w_k) = 0$  for any  $2 \leq k \leq c_j$ . Hence the trace of  $fh$  is equal to 1. The non-degenerate nature of  $\alpha$  implies the existence of a dual basis  $\{e_1, \dots, e_{c_j^2}\}$  of  $M_{c_j}(\mathbb{C})$  such that  $\alpha(g_k, e_l) = \delta_{kl}$  for any  $1 \leq k, l \leq c_j^2$ . Thus we obtain

$$g = \sum_{k=1}^{c_j^2} \alpha(g, g_k) e_k = \sum_{k=1}^{c_j^2} \mathrm{trace}(gg_k) e_k$$

for any  $g \in M_{c_j}(\mathbb{C})$ . We deduce

$$H_j \subset \left\{ \sum_{k=1}^{c_j^2} t_k e_k : t_k \in \tau(H_j) \right\}$$

where  $\tau(H_j)$  is the set of traces of elements of  $H_j$ . The lack of hyperbolic elements implies  $\tau(H_j) \subset \overline{\mathbb{B}_{c_j}^1}$ . In particular  $H_j$  is bounded and then relatively compact in  $\mathrm{GL}(V_j/V_{j-1})$  for any  $1 \leq j \leq r$ .  $\square$

**Lemma 5.2.** *The group  $\overline{H}_j$  consists of semisimple (i.e. diagonalizable) non-hyperbolic transformations for any  $1 \leq j \leq r$ .*

*Proof.* Fix  $1 \leq j \leq r$ . Since the spectrum of a matrix varies continuously, we obtain  $\mathrm{spec}(A) \subset \mathbb{S}^1$  for any  $A \in \overline{H}_j$ . Fix  $A \in \overline{H}_j$ . Consider the multiplicative Jordan decomposition  $A = A_s A_u$  as the product of a semisimple matrix  $A_s$  and a unipotent matrix  $A_u$  that commute. Since  $\mathrm{spec}(A) = \mathrm{spec}(A_s)$  and  $A_s$  is diagonalizable, the group  $\langle A_s \rangle$  is relatively compact. Hence  $\langle A_u \rangle$  is relatively compact. Let us show  $A_u = Id$  by contradiction. Otherwise the Jordan normal form theorem implies the existence of linearly independent vectors  $v, w$  such that  $A_u v = v + w$  and  $A_u w = w$ . Since  $A_u^j v = v + jw$  for  $j \in \mathbb{Z}$  the group  $\langle A_u \rangle$  is non-relatively compact and we obtain a contradiction.  $\square$

Consider a free subgroup  $\langle a, b \rangle$  on two generators of  $\mathrm{GL}(n, \mathbb{C})$  with  $a, b$  close to  $Id$ . The next result has two functions: namely showing that  $\langle a, b \rangle$  is non-0-pseudo-solvable (and then non- $p$ -pseudo-solvable for

any  $p \in \mathbb{N} \cup \{0\}$ ) and finding non-abelian free subgroups of  $\langle a, b \rangle$  with free generators even closer to  $Id$ .

**Lemma 5.3.** *Let  $H$  be a free group on  $\{a, b\}$  and  $\mathcal{S} = \{a, b, a^{-1}, b^{-1}\}$ . Then  $\mathcal{S}_0(2k)$  contains 2 elements that are free generators of a free group for any  $k \geq 0$ . Moreover the set  $\cup_{j \geq 0} \mathcal{S}_0(j)$  is infinite.*

*Proof.* Fix  $k \geq 0$ . We want to prove that in  $\mathcal{S}_0(2k)$  there is a word  $\alpha_{f,k}$  that in reduced form has length  $4^{2k}$  and  $f$  as first and also as last letter for any  $f \in \mathcal{S}$ . The result is obvious for  $k = 0$ . Let us show that if it holds for  $k$  then so it does for  $k + 1$ . We have that

$$[\alpha_{a,k}, \alpha_{b^{-1},k}] = a \dots b \text{ and } [\alpha_{a^{-1},k}, \alpha_{b,k}] = a^{-1} \dots b^{-1}$$

are words of length  $4^{2k+1}$  in  $\mathcal{S}_0(2(k+1))$ . We can define

$$\alpha_{a,k+1} = [[\alpha_{a,k}, \alpha_{b^{-1},k}], [\alpha_{a^{-1},k}, \alpha_{b,k}]] = a \dots ba^{-1} \dots b^{-1}b^{-1} \dots a^{-1}b \dots a.$$

The word  $\alpha_{a,k+1}$  belongs to  $\mathcal{S}_0(2(k+1))$  and has length  $4^{2(k+1)}$ . The words  $\alpha_{a^{-1},k+1}$ ,  $\alpha_{b,k+1}$  and  $\alpha_{b^{-1},k+1}$  are defined analogously.

It is clear that  $\alpha_{a,k}$  and  $\alpha_{b,k}$  are free generators of a free group on two elements for any  $k \geq 0$ . Moreover  $\cup_{j \geq 0} \mathcal{S}_0(j)$  is infinite since it contains reduced words of arbitrarily high length.  $\square$

The next result is a simple exercise; it is a consequence of the compactness of  $(\mathbb{S}^1)^l$ .

**Lemma 5.4.** *Consider  $\lambda_1, \dots, \lambda_l \in \mathbb{S}^1$ . Then there exists an increasing sequence  $(n_k)_{k \geq 1}$  of natural numbers such that  $\lim_{k \rightarrow \infty} \lambda_j^{n_k} = 1$  for any  $1 \leq j \leq l$ .*

*end of the proof of Theorem 5.1.* Let us remind the reader that a subgroup of  $GL(n, \mathbb{C})$  is either virtually solvable or it contains a non-abelian free group by the Tits alternative [19]. Therefore there exist  $A, B \in H$  that are free generators of a free group on two elements. Lemmas 5.2 and 5.4 imply the existence of a increasing sequence  $(n_k)_{k \geq 1}$  of natural numbers such that  $A_{|V_j/V_{j-1}}^{n_k}$  and  $B_{|V_j/V_{j-1}}^{n_k}$  tend to the identity map when  $k \rightarrow \infty$  for any  $1 \leq j \leq r$ .

Given a matrix  $D$  we denote by  $D_{I,J}$  the minor that corresponds to the rows with index  $I$  and the columns with index  $J$ . We define the diagonal minors  $D_j = D_{\{d_{j-1}+1, \dots, d_j\}, \{d_{j-1}+1, \dots, d_j\}}$  for any  $1 \leq j \leq r$ . Consider a base  $\mathcal{B} = \{v_1, \dots, v_n\}$  of  $\mathbb{C}^n$  such that  $\{v_1, \dots, v_{d_l}\}$  is a base of  $V_l$  for any  $1 \leq l \leq r$ . We identify  $A$  and  $B$  with its matrices in the basis  $\mathcal{B}$ . They are block upper triangular matrices. The condition  $\lim_{k \rightarrow \infty} A_{|V_j/V_{j-1}}^{n_k} = Id$  implies  $\lim_{k \rightarrow \infty} (A^{n_k})_j = Id$ . We define the matrix  $C^s$  for  $s \in \mathbb{R}^+$  such that  $C_{j,j}^s = s^l$  if  $d_l < j \leq d_{l+1}$  and  $C_{j,k}^s = 0$



if  $j \neq k$ . We obtain  $((C^s)^{-1}A^{n_k}C^s)_j = A_j^{n_k}$  and  $((C^s)^{-1}B^{n_k}C^s)_j = B_j^{n_k}$  for all  $s > 0$ ,  $1 \leq j \leq r$  and  $k \geq 1$  whereas all the coefficients of the matrix  $(C^s)^{-1}B^{n_k}C^s$  outside of the diagonal minors tend to 0 when  $s \rightarrow 0$ . Hence by considering  $k \in \mathbb{N}$  big enough and then  $s > 0$  small enough we obtain matrices  $\tilde{A} = (C^s)^{-1}A^{n_k}C^s$  and  $\tilde{B} = (C^s)^{-1}B^{n_k}C^s$  arbitrarily close to the identity. Moreover  $\tilde{A}$  and  $\tilde{B}$  are free generators of the group  $\langle \tilde{A}, \tilde{B} \rangle$ . Let  $W$  be the neighborhood of  $Id$  provided by Remark 2.2 for  $\mathrm{GL}(n, \mathbb{C})$ . We can suppose  $\tilde{A}, \tilde{B}, \tilde{A}^{-1}\tilde{B}^{-1} \in W$ .

We define  $\mathcal{S} = \{\tilde{A}, \tilde{B}, \tilde{A}^{-1}\tilde{B}^{-1}\}$ . The elements in  $\mathcal{S}_0(k)$  are contained in  $\{D \in \mathrm{GL}(n, \mathbb{C}) : \|D - Id\| < \epsilon/2^{2^k-1}\}$  (cf. Remark 2.2). Lemma 5.3 implies the existence of free groups on two elements of  $(C^s)^{-1}HC^s$  arbitrarily close to  $Id$ . Therefore there exist free groups on two elements of  $H$  arbitrarily close to  $Id$ .  $\square$

**5.2. Groups with non-solvable connected component of  $Id$ .** In our quest for free groups we consider now subgroups  $H$  of  $\mathrm{GL}(n, \mathbb{C})$  such that  $\overline{H}_0$  is non-solvable.

**Definition 5.3.** We say that a connected Lie group  $T$  is topologically perfect if  $T = \overline{[T, T]}$ .

Next we provide the statements of two theorems by Breuillard and Gelander that we use to find free groups in a neighborhood of  $Id$ . In particular Theorem 5.3 is particularly interesting since it localizes the free generators of the free group

**Theorem 5.2.** [2] *Let  $T$  be a topologically perfect real Lie group. There exists a neighborhood  $\Omega$  of  $Id$ , where  $\exp^{-1}$  is a diffeomorphism from  $\Omega$  to a neighborhood of 0 in  $L(T)$ , such that given  $f_1, \dots, f_m \in \Omega$ , the group  $\langle f_1, \dots, f_m \rangle$  is dense in  $T$  if  $\log f_1, \dots, \log f_m$  generates  $L(T)$ . The neighborhood  $\Omega$  can be considered arbitrarily small.*

**Theorem 5.3.** [3] *Let  $k$  be a characteristic 0 local field. Let  $T = \langle f_1, \dots, f_m \rangle$  be a subgroup of  $\mathrm{GL}(n, k)$  that contains no open solvable subgroup. Then given any neighborhood  $\Omega$  of  $Id$  in  $T$ , there exist  $g_j \in \Omega f_j \Omega$  for any  $1 \leq j \leq m$  such that  $g_1, \dots, g_m$  are free generators of a free dense subgroup of  $T$ .*

The algebraic properties of a linear group and its topological closure coincide. Next lemma provides a particular case of the previous principle.

**Lemma 5.5.** *Given a subgroup  $T$  of  $\mathrm{GL}(n, \mathbb{C})$  we have  $T^{(j)} \subset \overline{T}^{(j)} \subset \overline{T^{(j)}}$  for any  $j \geq 0$ . In particular we deduce that  $\ell(T)$  is equal to  $\ell(\overline{T})$  and that  $T$  is solvable if and only if  $\overline{T}$  is solvable.*

*Proof.* We have  $\overline{T}^{(1)} \subset \overline{T^{(1)}}$ . By recurrence we obtain  $T^{(j)} \subset \overline{T}^{(j)} \subset \overline{T^{(j)}}$  for any  $j \geq 0$ . In particular we obtain  $\ell(T) = \ell(\overline{T})$ .  $\square$

Next we introduce and prove the main result of this section.

**Theorem 5.4.** *Let  $H$  be a subgroup of  $\mathrm{GL}(n, \mathbb{C})$  such that  $\overline{H}_0$  is non-solvable. Then given any neighborhood  $U$  of  $Id$  in  $H$ , there exists free generators  $f, g \in H \cap U$  of a free group on two elements.*

*Proof.* We denote  $H_0^\circ = \overline{H}_0$  and  $H_{j+1}^\circ = \overline{[H_j^\circ, H_j^\circ]}$  for  $j \geq 0$ . The construction of  $H_j^\circ$  implies  $\overline{(\overline{H}_0 \cap H)^{(j)}} \subset H_j^\circ$  for any  $j \geq 0$ . Let us show that  $H_j^\circ$  is contained in  $\overline{(\overline{H}_0 \cap H)^{(j)}}$  for  $j \geq 0$ . It is obvious for  $j = 0$ . Suppose it holds for  $j \geq 0$ . We have

$$H_{j+1}^\circ \subset \overline{[\overline{(\overline{H}_0 \cap H)^{(j)}}, \overline{(\overline{H}_0 \cap H)^{(j)}}]} \subset \overline{(\overline{H}_0 \cap H)^{(j+1)}} = \overline{(\overline{H}_0 \cap H)^{(j+1)}}$$

by Lemma 5.5. We obtain  $H_j^\circ \subset \overline{(\overline{H}_0 \cap H)^{(j)}}$  for any  $j \geq 0$  by induction on  $j$ . Thus  $H_j^\circ = \overline{(\overline{H}_0 \cap H)^{(j)}}$  holds for any  $j \geq 0$ .

Since  $\overline{H}_0 \cap H$  is dense in  $\overline{H}_0$ , the group  $\overline{H}_0 \cap H$  is non-solvable by Lemma 5.5. A simple consequence is that  $H_j^\circ$  is never the trivial group for any  $j \geq 0$ .

The group  $H_0^\circ$  is connected. Moreover if  $H_j^\circ$  is connected then  $[H_j^\circ, H_j^\circ]$  and  $H_{j+1}^\circ$  are connected too for  $j \geq 0$ . We deduce that  $H_j^\circ$  is connected for any  $j \geq 0$ . We obtain a decreasing sequence  $H_0^\circ \supset H_1^\circ \supset H_2^\circ \supset \dots$  of connected Lie groups. Given  $j \geq 0$  either we have  $H_j^\circ = H_{j+1}^\circ$  or  $\dim H_j^\circ < \dim H_{j+1}^\circ$ . Hence there exists  $j_0 \geq 0$  such that  $\{Id\} \neq H_{j_0}^\circ = H_{j_0+1}^\circ$ . By replacing  $H$  with  $(\overline{H}_0 \cap H)^{(j_0)}$  we can suppose that  $H$  is a non-solvable group such that  $\overline{H}$  is topologically perfect.

Let  $\Omega$  be a neighborhood of  $Id$  for  $\overline{H}$  as provided by Theorem 5.2. We can suppose  $\Omega \subset U$ . It is clear that there exist  $f_1, \dots, f_m \in \Omega \cap H$  such that  $\overline{H} = \overline{J}$  where  $J = \langle f_1, \dots, f_m \rangle$ . Moreover this can be achieved for  $m = \dim \overline{H}$ .

Let us show that  $J$  does not contain any open solvable subgroup. Otherwise there exists a neighborhood  $V$  of  $Id$  in  $\mathrm{GL}(n, \mathbb{C})$  such that  $\langle J \cap V \rangle$  is solvable. The group  $\overline{\langle J \cap V \rangle}$  is solvable by Lemma 5.5. Since such a group contains  $\overline{\langle J \cap V \rangle}$ , we deduce that  $\langle \overline{H} \cap V \rangle$  is solvable. A connected Lie group is generated by any of its neighborhoods of the identity, hence  $\overline{H}$  is solvable and we obtain a contradiction.

Theorem 5.3 implies that there exist  $g_1, \dots, g_m \in J$  arbitrarily close to  $f_1, \dots, f_m$  respectively such that  $g_1, \dots, g_m$  are free generators

of a free dense subgroup of  $J$ . We can suppose  $g_1, \dots, g_m \in U$ . It is clear that  $m \geq 2$  since  $\langle g_1, \dots, g_m \rangle$  is non-solvable by Lemma 5.5.  $\square$

**5.3. Irreducible and non-discrete groups.** We consider conditions on a linear group  $H$  forcing  $\overline{H}_0$  to be non-solvable. Indeed we require  $H$  and all its finite index subgroups to be irreducible and the group induced by  $H$  in  $\mathrm{PGL}(n, \mathbb{C})$  to be discrete (cf. Theorem 5.6). The next result corresponds to one of the cases in Theorem 1 and it is also an intermediate result to show Theorem 5.6.

**Theorem 5.5.** *Let  $H$  be a subgroup of  $\mathrm{GL}(n, \mathbb{C})$  that is not virtually reducible. Suppose further that  $\overline{H}_0$  is not contained in  $\mathbb{C}^* \mathrm{Id}$ . Then given any neighborhood  $U$  of  $\mathrm{Id}$  in  $H$ , there exist  $f, g \in H \cap U$  such that  $f, g$  are free generators of the free group  $\langle f, g \rangle$ .*

*Proof.* It suffices to show that the hypotheses imply that  $\overline{H}_0$  is non-solvable and then to apply Theorem 5.4.

Suppose  $\overline{H}_0$  is solvable. We define

$$S = \{v \in \mathbb{C}^n : \exists \lambda_{A,v} \in \mathbb{C} \text{ such that } Av = \lambda_{A,v}v \quad \forall A \in \overline{H}_0\}$$

of vectors that are eigenvectors for all transformations in  $\overline{H}_0$ . The set  $S$  is a finite union  $V_1, \dots, V_m$  such that  $V_1 + \dots + V_m$  is a direct sum. Moreover  $A|_{V_j}$  is a multiple of the identity map for all  $A \in \overline{H}_0$  and  $1 \leq j \leq m$ . Let us remark that  $S \neq \{0\}$  since all elements of  $\overline{H}_0$  have a common eigenvector by Lie-Kolchin theorem (cf. [17][p. 38, Theorem 5.1\*]). The group  $\overline{H}_0$  is normal in  $\overline{H}$ , hence the elements of  $\overline{H}$  permute the subspaces  $V_1, \dots, V_m$ . There exists a finite index normal subgroup  $J$  of  $H$  such that  $A(V_j) = V_j$  for any  $1 \leq j \leq m$ . Notice that the situation  $m = 1$  and  $V_1 = \mathbb{C}^n$  is impossible since then  $\overline{H}_0 \subset \mathbb{C}^* \mathrm{Id}$ . We deduce that  $J$  is reducible and then  $H$  is virtually reducible, obtaining a contradiction.  $\square$

**Theorem 5.6.** *Let  $H$  be a subgroup of  $\mathrm{GL}(n, \mathbb{C})$  that is not virtually reducible. Suppose that the group induced by  $H$  in  $\mathrm{PGL}(n, \mathbb{C})$  is non-discrete. Then given any neighborhood  $U$  of  $\mathrm{Id}$  in  $\mathrm{GL}(n, \mathbb{C})$ , there exist  $A, B \in H \cap U$  such that  $A, B$  are free generators of the free group  $\langle A, B \rangle$ . In particular  $\overline{H}_0$  is non-virtually solvable.*

*Proof.* Consider the natural map  $\Lambda : \mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{PGL}(n, \mathbb{C})$ . We denote  $J = \Lambda^{-1}(\Lambda(H))$ . The group  $J$  is not virtually reducible. Even if a priori  $H$  can be strictly contained in  $J$ , its derived groups  $H^{(1)}$  and  $J^{(1)}$  coincide.

Since  $\Lambda(H)$  is non-discrete, the group  $\overline{J}$  has elements arbitrarily close to  $\mathrm{Id}$  that do not belong to  $\mathbb{C}^* \mathrm{Id}$ . We deduce that  $\overline{J}_0$  is not

contained in  $\mathbb{C}^*Id$ . There are elements  $C, D \in J \cap U$  that are free generators of a free group on two elements by Theorem 5.5. We can even suppose  $C, D, C^{-1}, D^{-1} \in W$  where  $W$  is the neighbourhood of  $Id$  in  $GL(n, \mathbb{C})$  provided by Remark 2.2. We define  $\mathcal{S} = \{C, D, C^{-1}, D^{-1}\}$ . The set  $\mathcal{S}_0(k)$  is contained in  $\{E \in GL(n, \mathbb{C}) : \|E - Id\| < \epsilon/2^{2^k-1}\}$  (cf. Remark 2.2) for any  $k \geq 0$ . By considering  $k > 0$  big enough we obtain free groups on two elements whose free generators are contained in  $U \cap \mathcal{S}_0(2k) \cap J$  (Lemma 5.3). Since  $\mathcal{S}_0(2k) \subset J^{(1)} = H^{(1)} \subset H$  we are done.  $\square$

We have already all the ingredients to prove the main theorem.

*Proof of Theorem 1.* The result is a consequence of Proposition 4.7 in Case (1). Hence we can always suppose that  $G^1 := G \cap \text{Diff}_1(\mathbb{C}^n, 0)$  is solvable.

Let us show that in the remaining cases, given  $\epsilon > 0$  there exist  $A, B \in j^1G$  such that  $\|A - Id\|_1 < \epsilon$ ,  $\|B - Id\|_1 < \epsilon$  and  $A, B$  are free generators of a free group on two elements.

Let us consider Case (2). It suffices to show that  $j^1G$  is non-virtually solvable by Theorem 5.1. Suppose  $j^1G$  is virtually solvable. The natural morphism  $G/G^1 \rightarrow j^1G$  is an isomorphism of groups. Hence  $G/G^1$  is virtually solvable and there exists a finite index normal solvable subgroup  $H/G^1$  of  $G/G^1$ . Since  $(G/G^1)/(H/G^1)$  is isomorphic to  $G/H$ , we deduce that  $H$  is a finite index normal subgroup of  $G$ . The solvability of  $H/G^1$  and  $G^1$  implies that  $H$  is solvable. Therefore  $G$  is virtually solvable and we obtain a contradiction.

The result for Cases (3), (4) and (5) is a consequence of applying Theorems 5.4, 5.5 and 5.6 respectively to  $j^1G$ .

Consider  $\delta > 0$  with  $\delta < 1/8$ . There exist  $\phi, \eta \in G$  such that  $\|D_0\phi^{\pm 1} - Id\|_1 < \delta/8$ ,  $\|D_0\eta^{\pm 1} - Id\|_1 < \delta/8$  and  $D_0\phi, D_0\eta$  are free generators of a free group on two elements by the previous discussion. In particular  $\phi$  and  $\eta$  are free generators of a free group on two elements. We denote  $\mathcal{S} = \{\phi, \phi^{-1}, \eta, \eta^{-1}\}$ .

The group  $\langle \phi, \eta \rangle$  is not 0-pseudo-solvable for  $\mathcal{S}$  by Lemma 5.3. The proof is completed by applying Proposition 3.2.  $\square$

*Remark 5.1.* Conditions (2), (4) and (5) of Theorem 1 imply condition (3). This is a consequence of Theorems 5.1, 5.5 and 5.6 for Cases (2), (4) and (5) respectively.

**5.4. Groups with hyperbolic elements.** We show Theorem 2 in this section. It is natural to try to restrict our study to subgroups  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  such that  $j^1G$  is as simple as possible. For instance it is useful to consider linear groups  $j^1G$  that share Zariski-closure with

all its finite index normal subgroups. We will obtain such reduction by using the following proposition.

**Proposition 5.1.** *Let  $H$  be a non-virtually solvable subgroup of  $\mathrm{GL}(n, \mathbb{C})$ . There exists a subgroup  $J$  of  $H$  such that*

- $J$  is non-virtually solvable.
- $\overline{L}^z = \overline{J}^z$  for any non-virtually solvable subgroup  $L$  of  $J$ .
- $\overline{J}^z$  is a Zariski-connected irreducible algebraic group.
- $J$  is the derived group of a subgroup  $K$  of  $H$  satisfying the first two properties (and then  $\overline{K}^z = \overline{J}^z$ ).

The proposition is based in a simple idea: among the Zariski-closures of non-virtually solvable subgroups of a non-virtually solvable subgroup  $H$  of  $\mathrm{GL}(n, \mathbb{C})$  there are always minimal subgroups.

*Proof.* Let us show first that there exists a group  $K$  satisfying the first two properties by contradiction. Otherwise there exists an infinite decreasing sequence  $H = H_0 \supset H_1 \supset H_2 \supset \dots$  of non-virtually solvable subgroups of  $H$  such that

$$(8) \quad \overline{H_0}^z \supsetneq \overline{H_1}^z \supsetneq \overline{H_2}^z \supsetneq \dots$$

Given  $j \geq 0$  we have either  $\dim \overline{H_{j+1}}^z < \overline{H_j}^z$  or  $\overline{H_{j+1}}^z$  has fewer connected components than  $\overline{H_j}^z$ . Since the number of connected components of an algebraic group is finite, we deduce that the sequence (8) does not exist. We obtain a contradiction.

We claim that  $\overline{K}^z$  is connected. We define  $K' = (\overline{K}^z)_0 \cap K$ . The group  $K'$  is a finite index normal subgroup of  $K$  and hence  $K'$  is not virtually solvable. Moreover we have  $\overline{K'}^z = (\overline{K}^z)_0$ . Since  $\overline{K'}^z = \overline{K}^z$  by construction of  $K$ , we deduce  $\overline{K}^z = (\overline{K}^z)_0$ . Notice that  $\overline{K}^z$  is a (connected) smooth algebraic set since  $\overline{K}^z$  is an algebraic group. Therefore  $\overline{K}^z$  is an irreducible algebraic set.

The Tits alternative implies that  $K$  contains a free group on two elements and then  $J := K^{(1)}$  contains non-abelian free groups. In particular  $J$  is non-virtually solvable. We obtain  $\overline{J}^z = \overline{K}^z$  by the second property. Given a non-virtually solvable subgroup  $L$  of  $J$  we have  $\overline{J}^z = \overline{L}^z$ , otherwise  $K$  does not satisfy the second condition.  $\square$

The groups  $K$  and  $J$  provided by Proposition 5.1 share Zariski-closure with all their finite index subgroups. Such property is interesting, since for instance the set of  $J$ -invariant vector subspaces of  $\mathbb{C}^n$  depends only on  $\overline{J}^z$ .

**Lemma 5.6.** *Let  $H$  be a subgroup of  $\mathrm{GL}(n, \mathbb{C})$  and  $V$  a vector subspace of  $\mathbb{C}^n$ . Then  $V$  is  $H$ -invariant if and only if  $V$  is  $\overline{H}^z$ -invariant.*

*Proof.* The necessary condition is obvious. Let  $V$  be a  $H$ -invariant subspace. The set  $L := \{A \in \mathrm{GL}(n, \mathbb{C}) : A(V) = V\}$  is an algebraic group containing  $H$ . Hence  $L$  contains  $\overline{H}^z$ .  $\square$

Let  $G$  be a subgroup of  $\mathrm{Diff}(\mathbb{C}^n, 0)$  such that  $j^1G$  is non-virtually solvable. In order to show that  $G$  does not satisfy the discrete orbits property, we can suppose that there are elements in  $j^1G$  whose spectrum is not contained in  $\mathbb{S}^1$  by Theorem 1. These elements do have either stable or unstable manifolds. Indeed we will obtain recurrent point in stable or unstable manifolds of certain hyperbolic diffeomorphisms.

Let  $A \in \mathrm{GL}(n, \mathbb{C})$ . The subspaces

$$V_A^s = \bigoplus_{\lambda \in \mathrm{spec}(A) \cap \mathbb{B}_1^1} \ker(A - \lambda Id)^n, \quad V_A^{cu} = \bigoplus_{\lambda \in \mathrm{spec}(A) \setminus \mathbb{B}_1^1} \ker(A - \lambda Id)^n$$

are the stable and the center-unstable manifolds of  $A$  respectively. Consider  $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$  such that  $D_0\phi = A$ . The stable manifold theorem (cf. [15][p. 27]) implies that there exists a manifold  $W_\phi^s$  containing the origin such that  $\phi(W_\phi^s) \subset W_\phi^s$  and  $T_0W_\phi^s = V_A^s$ . Moreover  $\phi|_{W_\phi^s}^p$  tends uniformly to 0 in  $W_\phi^s$  when  $p \rightarrow \infty$  and the germ of  $W_\phi^s$  at the origin is uniquely determined. Analogously we define the unstable manifold  $W_\phi^u$ , it is equal to  $W_{\phi^{-1}}^s$ .

**Definition 5.4.** Let  $\pi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbf{CP}^{n-1}$  be the map associating to each vector its class in the projective space. Given a vector subspace  $V$  of  $\mathbb{C}^n$  we denote  $[V] = \pi(V \setminus \{0\})$ .

Next we show a irreducibility type property for actions on stable manifolds.

**Proposition 5.2.** *Let  $H$  be a non-virtually solvable subgroup of  $\mathrm{GL}(n, \mathbb{C})$ . Let  $K$  and  $J$  be the subgroups provided by Proposition 5.1. Consider  $A \in J$ . There is no a non-trivial  $J$ -invariant vector subspace of  $V_A^s$ . In particular given any  $[v] \in [V_A^s]$ , there exists  $B \in J$  such that  $[Bv] \notin [V_A^s] \cup [V_A^{cu}]$ .*

*Proof.* We denote  $V^s = V_A^s$  and  $V^{cu} = V_A^{cu}$ . Let  $0 \neq V \subset V^s$  be a  $J$ -invariant vector space. Proposition 5.1 implies  $\overline{J}^z = \overline{K}^z$ . Since  $V$  is  $J$ -invariant, it is also  $K$ -invariant by Lemma 5.6. The property  $J = K^{(1)}$  implies  $\det(B|_V) = 1$  for any  $B \in J$ . Since  $V \subset V^s$  this contradicts  $\mathrm{spec}(A|_V) \subset \mathrm{spec}(A|_{V^s}) \subset \{z \in \mathbb{C} : |z| < 1\}$ . Thus any  $J$ -invariant vector subspace of  $V^s$  is trivial.

Let

$$J_s = \{B \in J : [Bv] \in [V^s]\} \text{ and } J_{cu} = \{B \in J : [Bv] \in [V^{cu}]\}.$$

The sets  $J_s$  and  $J_{cu}$  are Zariski-closed in  $J$ . We claim  $J \not\subset J_s \cup J_{cu}$ . It suffices to show  $J \not\subset J_s$  and  $J \not\subset J_{cu}$  since  $\overline{J}^z$  is an irreducible algebraic set. The latter property is obvious since  $Id \notin J_{cu}$ .

Let us show  $J \not\subset J_s$  by contradiction. Otherwise  $Bv$  belongs to  $V^s$  for any  $B \in J$ . Therefore the linear subspace  $V$  generated by  $\{Bv : B \in J\}$  is contained in  $V^s$ . Moreover by construction  $V$  is  $J$ -invariant, contradicting the first part of the proof.  $\square$

*Proof of Theorem 2.* Suppose that  $G$  is not virtually solvable. We have

$$0 \rightarrow G \cap \text{Diff}_1(\mathbb{C}^n, 0) \rightarrow G \rightarrow j^1 G \rightarrow 0.$$

Suppose  $G \cap \text{Diff}_1(\mathbb{C}^n, 0)$  is non-solvable. Then  $G$  does not have the discrete orbits property since  $G \cap \text{Diff}_1(\mathbb{C}^n, 0)$  has (lots of) recurrent points in any neighborhood of the origin by Theorem 1 (1). Suppose from now on that  $G \cap \text{Diff}_1(\mathbb{C}^n, 0)$  is solvable. Since  $G$  is non-virtually solvable, the group  $H := j^1 G$  is non-virtually solvable.

Consider the groups  $J$  and  $K$  provided by Proposition 5.1. We define  $\tilde{G} = \{\phi \in G : j^1 \phi \in J\}$ . Suppose that  $\text{spec}(A) \subset \mathbb{S}^1$  for any  $A \in J$ . Since  $\tilde{G}$  is non-virtually solvable,  $G$  does not hold the discrete orbits property by Theorem 1 (2). We can suppose that there exists  $A \in J$  such that  $\text{spec}(A) \cap \mathbb{B}_1^1 \neq \emptyset$ . Consider an element  $\phi \in \tilde{G}$  such that  $j^1 \phi = A$ . We obtain  $\dim W_\phi^s \geq 1$ . Let  $D^s$  be a closed fundamental domain of  $\phi$  restricted to  $W_\phi^s \setminus \{0\}$ . We can extend  $D^s$  to a fundamental domain  $M^s$  for  $\phi$  restricted to a neighborhood of  $D^s$ .

Let  $q \in D^s$ . The sequence  $(\phi^p(q))_{p \geq 1}$  tends to 0 when  $p \rightarrow \infty$ . Up to consider a subsequence  $(p_k)_{k \geq 1}$  we can suppose that  $([\phi^{p_k}(q)])_{k \geq 1}$  converges to a direction  $\ell$  in  $[V_A^s]$  when  $k \rightarrow \infty$ . There exists  $B \in J$  such that  $[B](\ell) \notin [V_A^s] \cup [V_A^{cu}]$  by Proposition 5.2. Consider  $\psi \in \tilde{G}$  such that  $j^1 \psi = B$ . We deduce that  $\lim_{k \rightarrow \infty} [(\psi \circ \phi^{p_k})(q)] = [B](\ell)$ . Since  $[B](\ell) \notin [V_\phi^{cu}]$ , all accumulation points of the sequence  $([A^{-p}][B](\ell))_{p \geq 1}$  belong to  $[V_\phi^s]$ . We deduce that there exists a sequence  $(m_k)_{k \geq 1}$  of natural numbers such that  $\lim_{k \rightarrow \infty} (\phi^{-m_k} \circ \psi \circ \phi^{p_k})(q) \in D^s$ . Moreover  $(\psi \circ \phi^{p_k})(q)$  does not belong to  $W_\phi^s$  for  $k \gg 1$  since  $[B](\ell) \notin [V_\phi^s]$ . Since  $W_\phi^s$  is  $\phi$ -invariant, the point  $(\phi^{-m_k} \circ \psi \circ \phi^{p_k})(q)$  does not belong to  $W_\phi^s$  for any  $k \gg 1$ . In particular  $\lim_{k \rightarrow \infty} (\phi^{-m_k} \circ \psi \circ \phi^{p_k})(q)$  contains infinitely many points of the  $G$ -orbit of  $q$  in every of its neighborhoods. Next lemma applied to  $F = D^s$  implies that there exists a recurrent point for the action of the pseudogroup  $\mathcal{P}$  induced by  $G$  where we suppose that  $\phi$  is defined in a neighborhood of  $W_\phi^s$  and we can consider any domain of definition for the elements of  $G \setminus \langle \phi \rangle$ .  $\square$



Given a set  $S$  we denote by  $\wp(S)$  the power set of  $S$ . Let  $\mathcal{P}$  be a pseudogroup of homeomorphisms defined in a topological space  $M$ . Consider a closed subset  $F$  of  $M$ . We consider the map  $\tau : F \rightarrow \wp(F)$  where  $\tau(q)$  is defined as the set of points  $q' \in F$  such that  $q'$  contains infinitely many points of the  $\mathcal{P}$ -orbit of  $q$  for any neighborhood of  $q'$ . The map extends naturally to a map  $\tau : \wp(F) \rightarrow \wp(F)$  by defining  $\tau(S) = \cup_{q \in S} \tau(q)$  for any  $S \in \wp(F)$ .

**Lemma 5.7.** *Let  $\mathcal{P}$  be a pseudogroup of homeomorphisms defined in a Hausdorff topological space  $M$ . Consider a compact subset  $F$  of  $M$ . Suppose that  $\tau(q) \neq \emptyset$  for any  $q \in F$ . Then there exists a  $\mathcal{P}$ -recurrent point  $q' \in F$  or in other words  $q' \in \tau(q')$ .*

*Proof.* Let us enumerate some simple properties of  $\tau$ :

- $\tau(q)$  is a non-empty closed subset of  $F$  for any  $q \in F$ .
- $\tau(\tau(q)) \subset \tau(q)$  for any  $q \in F$ .

Let  $S$  be the subset of  $\wp(F)$  consisting of non-empty closed subsets  $T$  of  $F$  such that  $\tau(T) \subset T$ . Consider the order  $(S, \supset)$  defined by the reverse inclusion. Given a chain  $C$  in  $S$  the set  $\cap_{T \in C} T$  is a closed subset such that  $T \supset \cap_{T' \in C} T'$  for any  $T \in C$ . It is non-empty since the elements of the chain are compact. We have  $\tau(\cap_{T \in C} T) \subset \cap_{T \in C} \tau(T) \subset \cap_{T \in C} T$  and then  $\cap_{T \in C} T$  is an upper bound of the chain  $C$ . Zorn's lemma implies that there exists a maximal element  $T_0$  in  $S$ . Notice that  $\tau(q)$  for  $q \in T_0$  is a non-empty closed subset of  $T_0$  that belongs to  $S$  by the second property above. Since  $T_0$  is minimal for the inclusion, we obtain  $\tau(q) = T_0$  for any  $q \in T_0$ . In particular  $q$  belongs to  $\tau(q)$  for any  $q \in T_0$ .  $\square$

The proof of Theorem 2 splits in two parts, namely the cases that are consequence of Theorem 1 and the “hyperbolic” case. The proof in the later case has similarities with the proof in dimension 2 by Rebelo and Reis [12]; for instance Lemma 5.7 is inspired in one of their constructions.

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